PHASE-RESETTING IN THE YAMADA MODEL OF A *Q*-SWITCHED LASER

PHASE RESPONSE FROM AN INDUCED PERTURBATION

JACOB NGAHA, NEIL G. R. BRODERICK, AND BERND KRAUSKOPF

NZMS/AMS/AUSTMS JOINT MEETING

10TH DECEMBER, 2024

Te Whai Ao Dodd-Walls Centre for Photonic and Quantum Technologies

University of Auckland Waipapa Taumata Rau New Zealand

STABLE Q-SWITCHED LASERS

Journal of the Optical Society of America B
OPTICAL PHYSICS

Excitability in an all-fiber laser with a saturable absorber section

Robert Otupiri,[1,*] Bruno Garbin,[2] Neil G. R. Broderick,[1] and Bernd Krauskopf[3,4]

All optical Q-switched laser based spiking neuron

Keshia Mekemeza-Ona, Baptiste Routier and Benoît Charbonnier*

Université Grenoble-Alpes, CEA, Leti, Grenoble, France

- Optical frequency combs and optical clocks need stability
- How do they return to equilibrium when perturbed?
- Q-switched lasers can be optical analogues to neurons
  - Optical neural networks

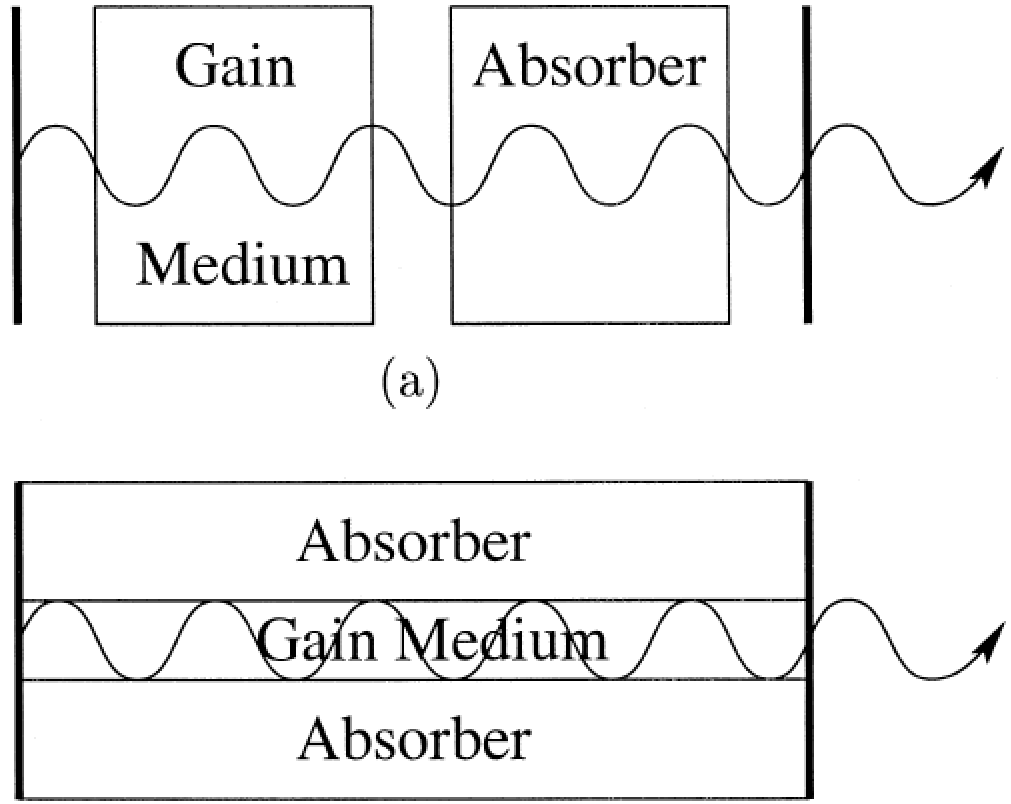
THE YAMADA MODEL

$$\dot{G} = \gamma\,(A - G - G\,I)$$
$$\dot{Q} = \gamma\,(B - Q - a\,Q\,I)$$
$$\dot{I} = (G - Q - 1)\,I$$

- G – Gain
- Q – Absorption
- I – Intensity

Parameters

- γ – Photon loss rate
- A – Pump current to gain
- B – Absorption coefficient
- a – Relative absorption vs. gain

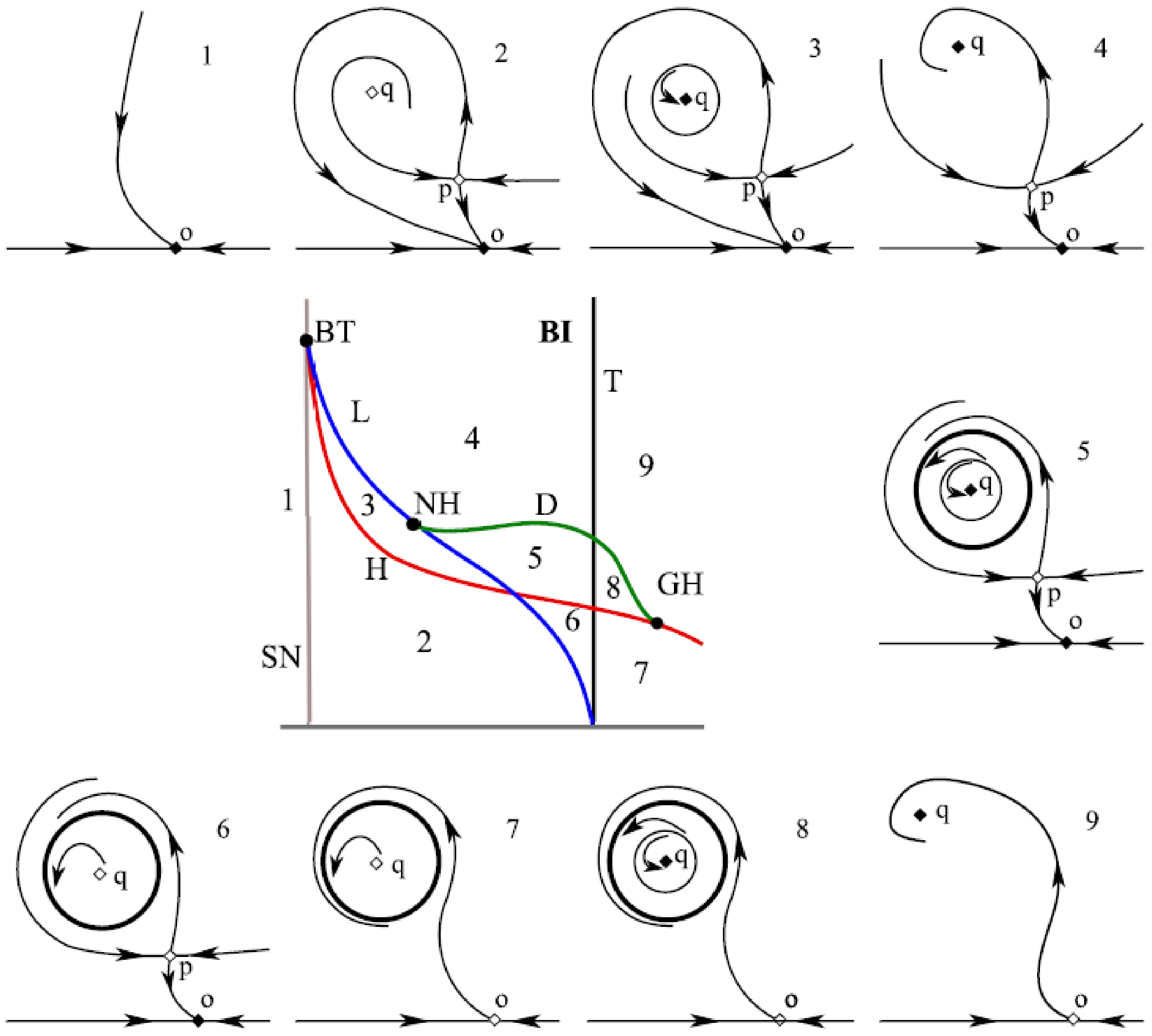


Taken from J. L. A. Dubbeldam and B. Krauskopf *"Self-pulsations of lasers with saturable absorber: Dynamics and bifurcations"*, Opt. Commun., **159** (4-6), 325 (1999).

THE YAMADA MODEL: BIFURCATION DIAGRAM

- Different dynamics split by bifurcations:
  - Hopf, homoclinic, saddle

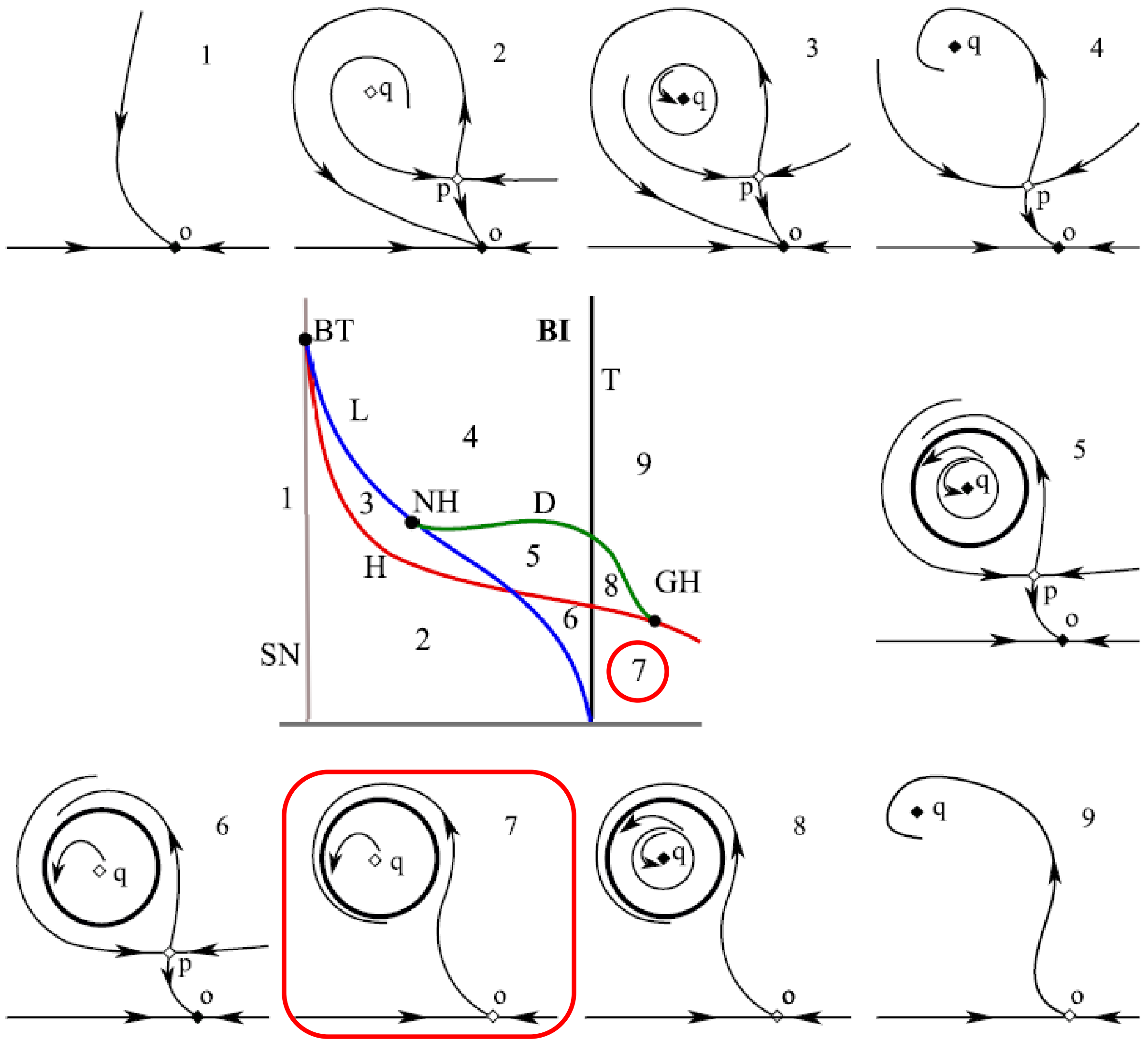
- Objects in phase space
  - *o* – Stable equilibrium (‘off state’)
  - *p* – Saddle with two unstable and one stable eigenvalues
  - *q* – Spiral source
  - Attracting periodic orbit
  - Saddle periodic orbit



Taken from R. Otupiri, B. Krauskopf, N. G. R. Broderick *“The Yamada Model for a Self-Pulsing Laser: Bifurcation Structure for Non Identical Decay Times of Gain and Absorber”*, Int. J. Bifurc. Chaos Appl. Sci. Eng., **30** (14) (2020).

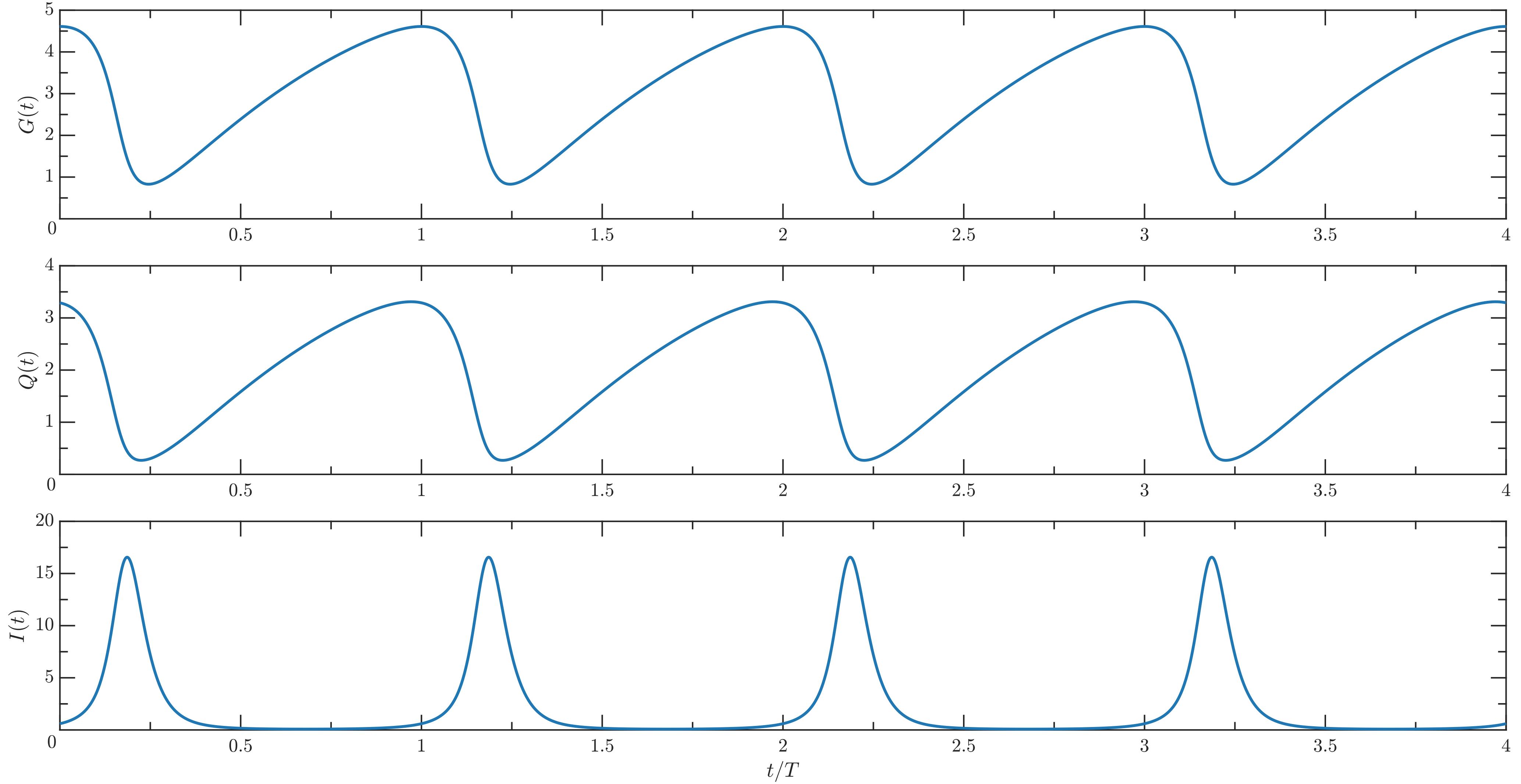
THE YAMADA MODEL: BIFURCATION DIAGRAM

- Different dynamics split by bifurcations:
  - Hopf, homoclinic, saddle
- Objects in phase space
  - *o* – Stable equilibrium (‘off state’)
  - *p* – Saddle with two unstable and one stable eigenvalues
  - *q* – Spiral source
  - Attracting periodic orbit
  - Saddle periodic orbit



Taken from R. Otupiri, B. Krauskopf, N. G. R. Broderick *“The Yamada Model for a Self-Pulsing Laser: Bifurcation Structure for Non Identical Decay Times of Gain and Absorber”*, Int. J. Bifurc. Chaos Appl. Sci. Eng., **30** (14) (2020).

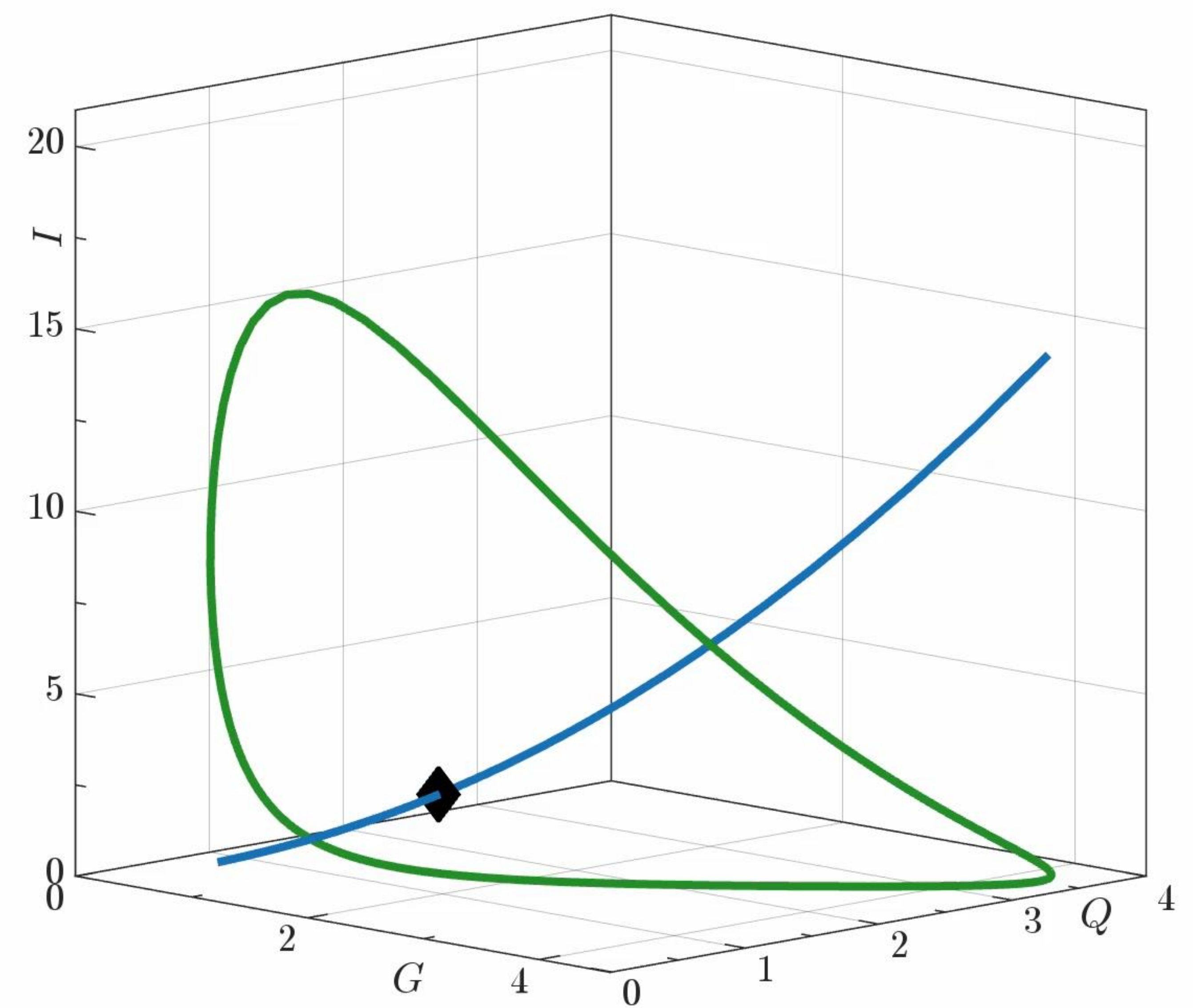
THE YAMADA MODEL: ATTRACTING PERIODIC ORBIT



$A = 7.3757$
$\gamma = 0.0354$
$B = 5.8$
$a = 1.8$

THE YAMADA MODEL: ATTRACTING PERIODIC ORBIT

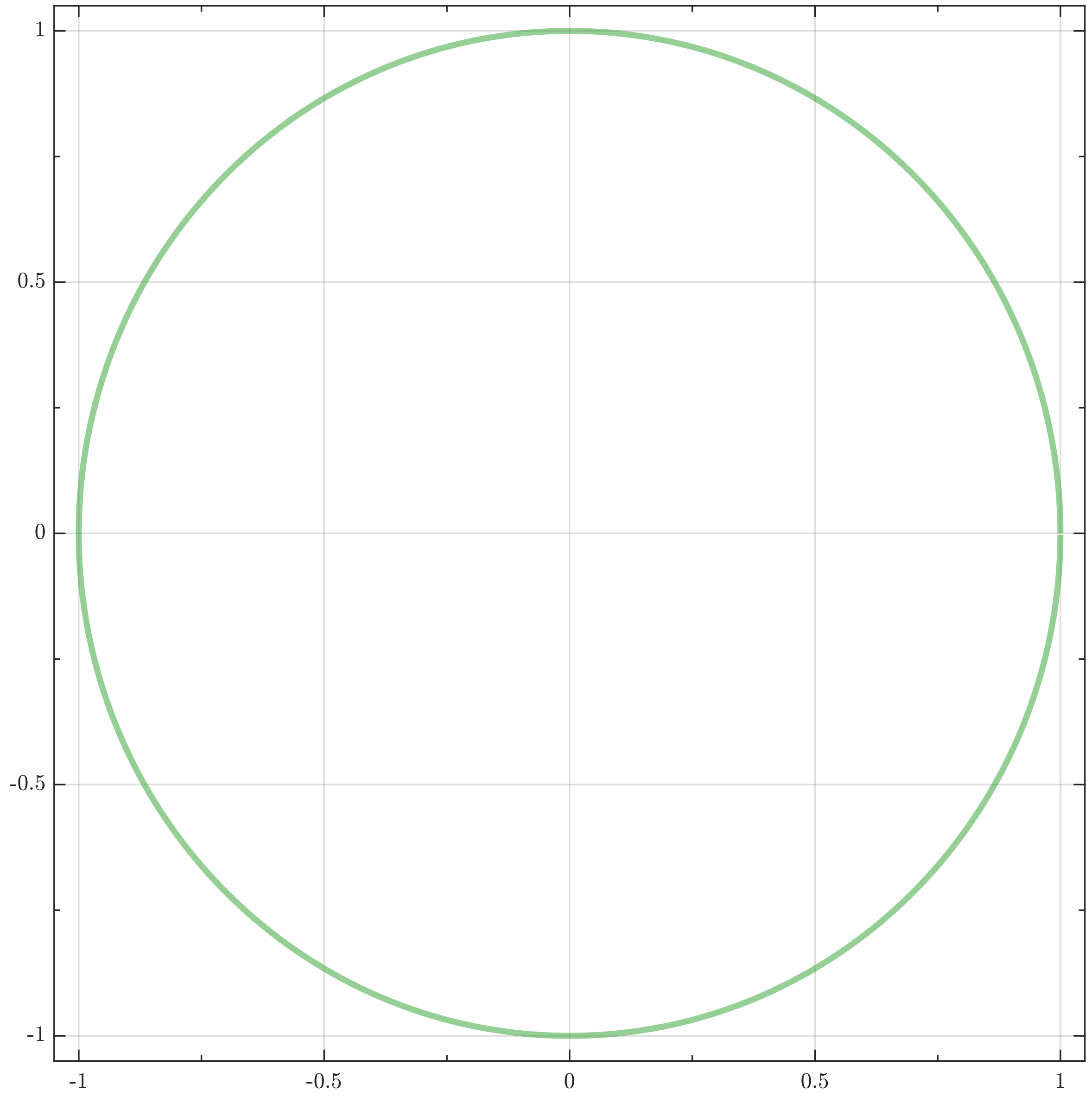
- Attracting periodic orbit Γ (green curve)
- Unstable stationary point / spiral source q (black diamond)
- One-dimensional stable manifold of the stationary point $W^s(q)$ (blue curve)



$A = 7.3757$
$\gamma = 0.0354$
$B = 5.8$
$a = 1.8$

PHASE-RESETTING

- Induced perturbation
  - A_{p} – amplitude
  - $d_{\mathrm{p}} = \left(\cos\theta_{\mathrm{p}}, \sin\theta_{\mathrm{p}}\right)$ – direction
  - θ_{old} – phase perturbation is applied
- When does the perturbed segment return?
  - θ_{new} – phase perturbation returns
- Boundary value problem (BVP)
  - Numerical continuation in AUTO and COCO

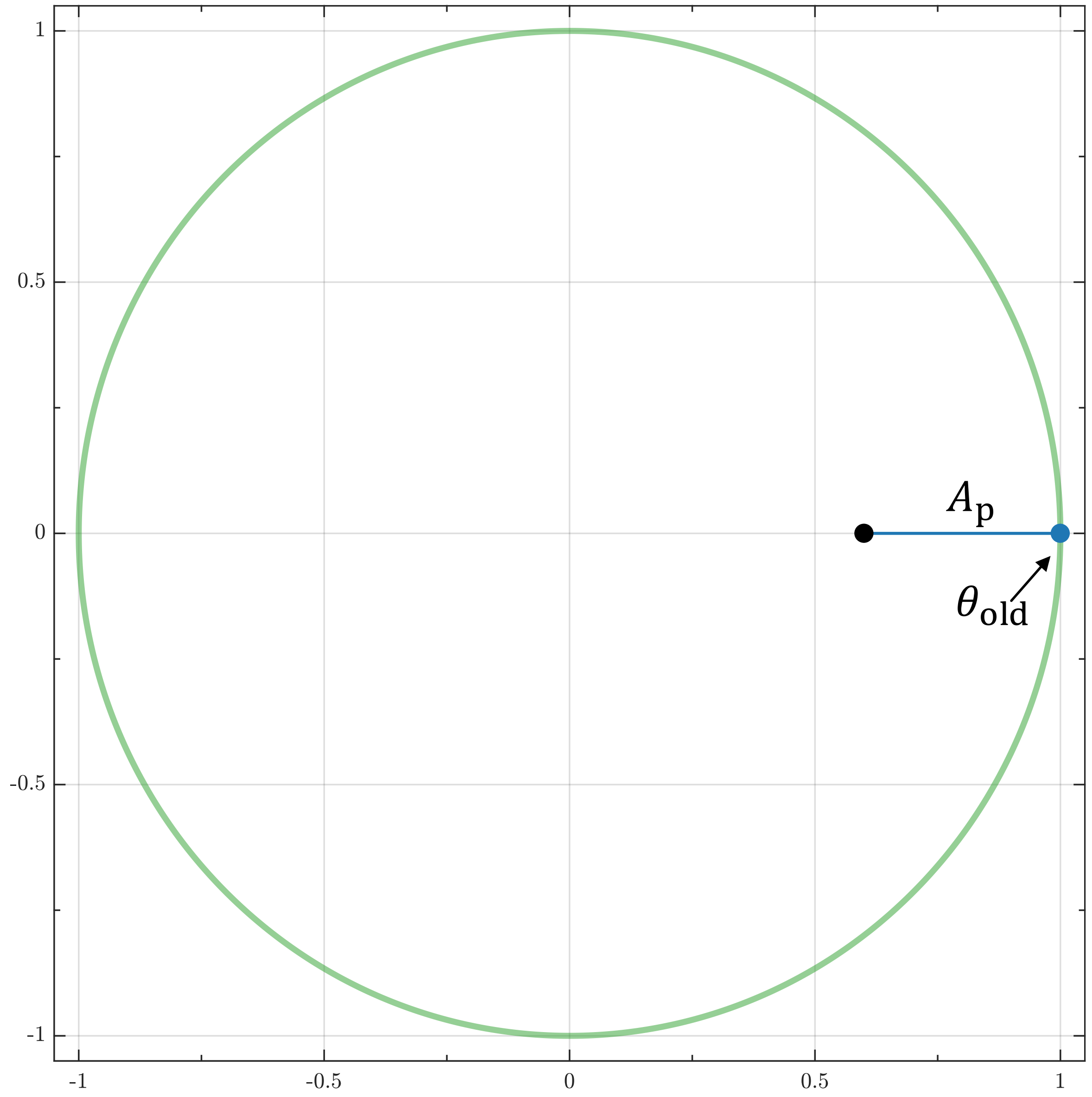


PHASE-RESETTING

- Induced perturbation
  - A_{p} – amplitude
  - $d_{\mathrm{p}} = (\cos\theta_{\mathrm{p}}, \sin\theta_{\mathrm{p}})$ – direction
  - θ_{old} – phase perturbation is applied

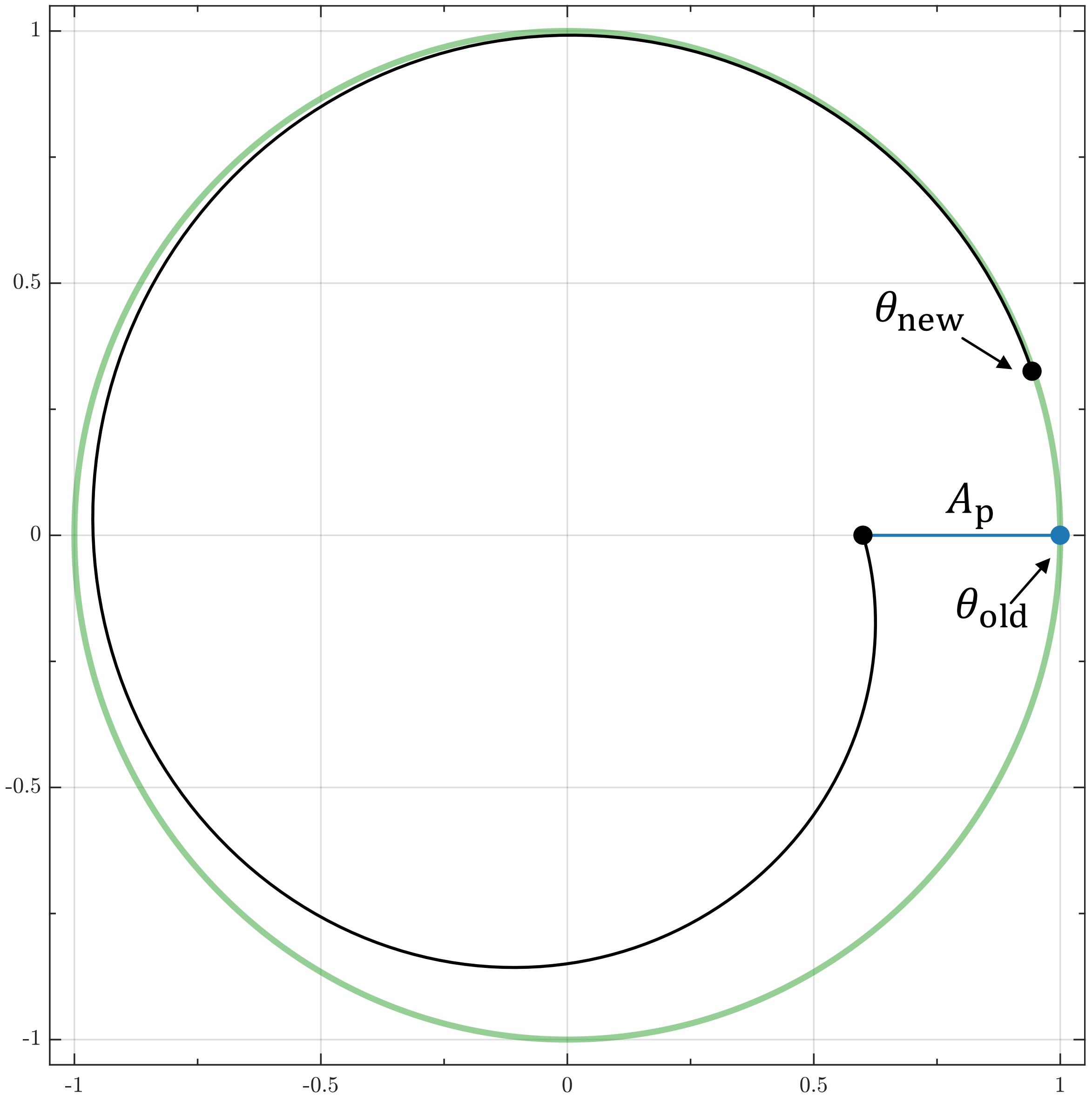
- When does the perturbed segment return?
  - θ_{new} – phase perturbation returns

- Boundary value problem (BVP)
  - Numerical continuation in AUTO and COCO



PHASE-RESETTING

- Induced perturbation
  - A_{p} – amplitude
  - $d_{\mathrm{p}} = (\cos\theta_{\mathrm{p}}, \sin\theta_{\mathrm{p}})$ – direction
  - θ_{old} – phase perturbation is applied
- When does the perturbed segment return?
  - θ_{new} – phase perturbation returns
- Boundary value problem (BVP)
  - Numerical continuation in AUTO and COCO



PHASE-RESETTING

- Induced perturbation
  - A_{p} – amplitude
  - $d_{\mathrm{p}} = (\cos\theta_{\mathrm{p}}, \sin\theta_{\mathrm{p}})$ – direction
  - θ_{old} – phase perturbation is applied

- When does the perturbed segment return?
  - θ_{new} – phase perturbation returns

- Boundary value problem (BVP)
  - Numerical continuation in AUTO and COCO

A Continuation Approach to Computing Phase Resetting Curves

Peter Langfield[1,2], Bernd Krauskopf[3], and Hinke M. Osinga[3(✉)]

Phase response to arbitrary perturbations: Geometric insights and resetting surfaces

Kyoung H. Lee[1], Neil G. R. Broderick[2], Bernd Krauskopf[1] and Hinke M. Osinga[1]

SIAM J. APPLIED DYNAMICAL SYSTEMS
Vol. 14, No. 3, pp. 1418–1453

Forward-Time and Backward-Time Isochrons and Their Interactions*

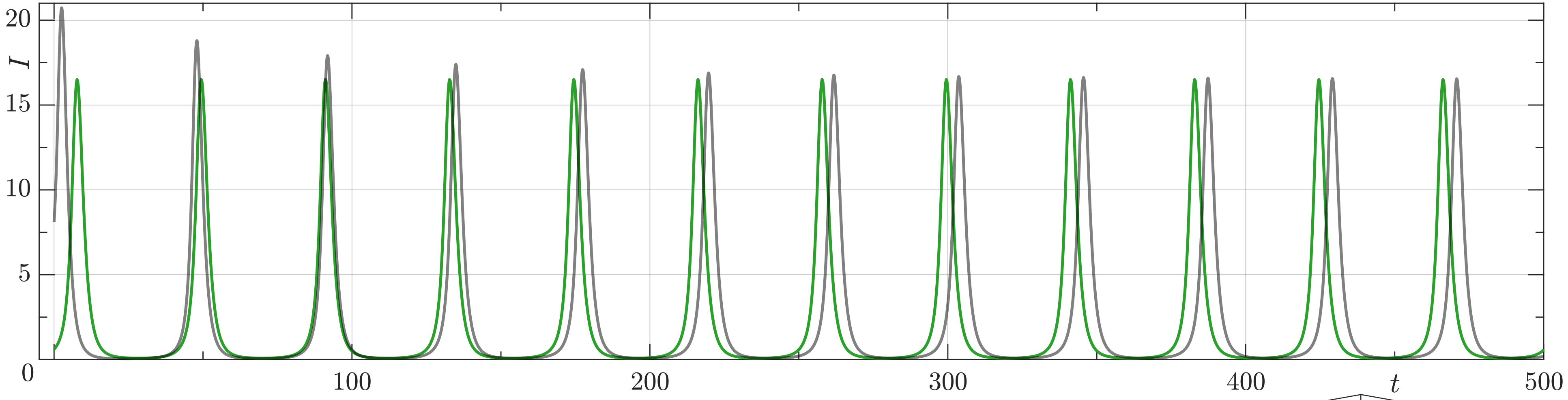
Peter Langfield†, Bernd Krauskopf†, and Hinke M. Osinga†

SIAM J. APPLIED DYNAMICAL SYSTEMS
Vol. 9, No. 4, pp. 1201–1228

Continuation-based Computation of Global Isochrons*

Hinke M. Osinga† and Jeff Moehlis‡

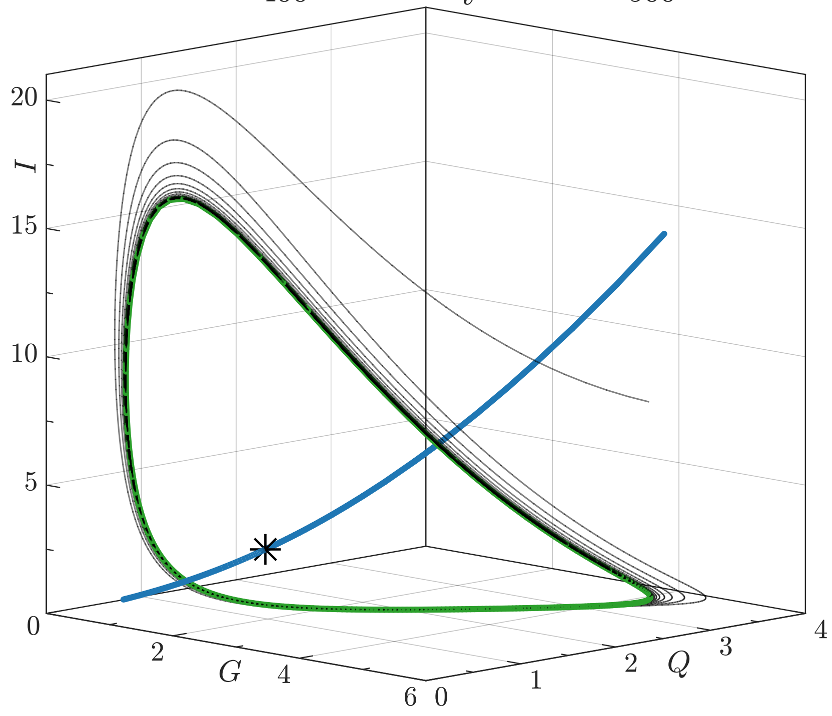
PHASE-RESETTING: YAMADA MODEL



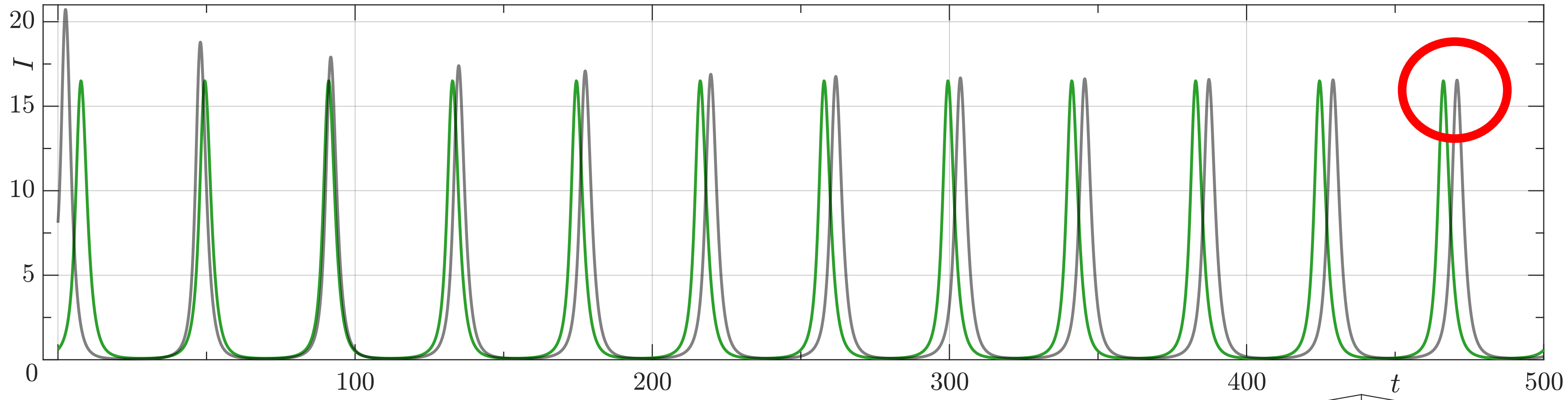
- Perturbations cause a phase shift (‘lag’) in intensity pulses.

- Phase difference ≈ $T_\Gamma \, \theta_{\text{new}}$

- Relationship between A_{p}, θ_{old}, and θ_{new}?



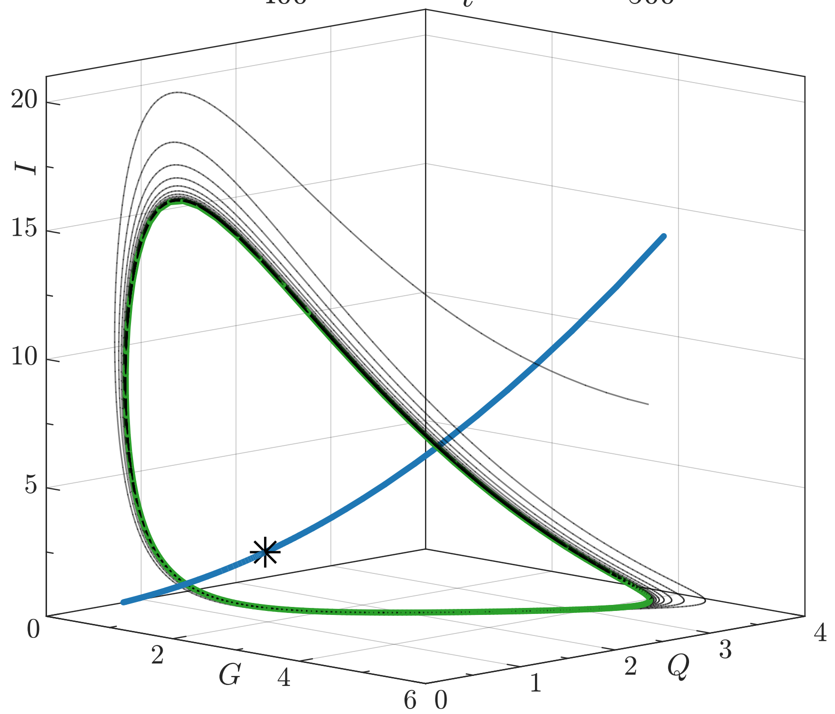
PHASE-RESETTING: YAMADA MODEL



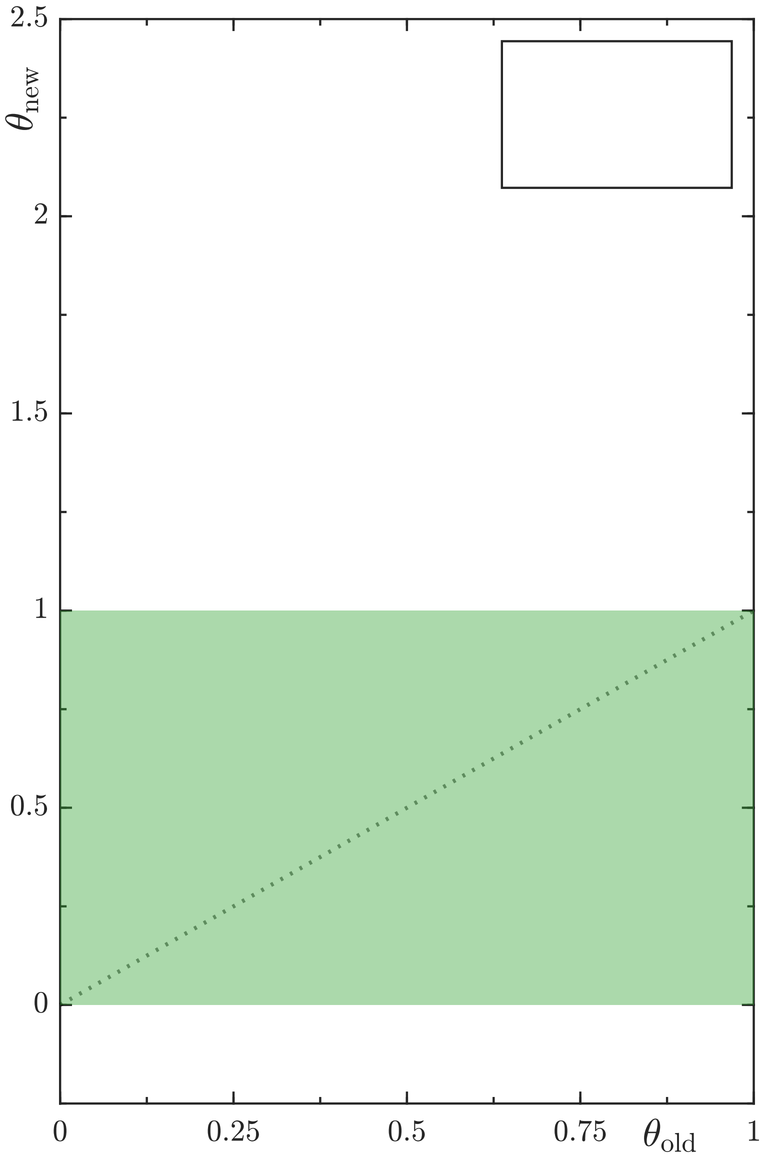
- Perturbations cause a phase shift (‘lag’) in intensity pulses.

- Phase difference ≈ $\theta_{\text{old}} - \theta_{\text{new}}$

- Relationship between A_{p}, θ_{old}, and θ_{new}?

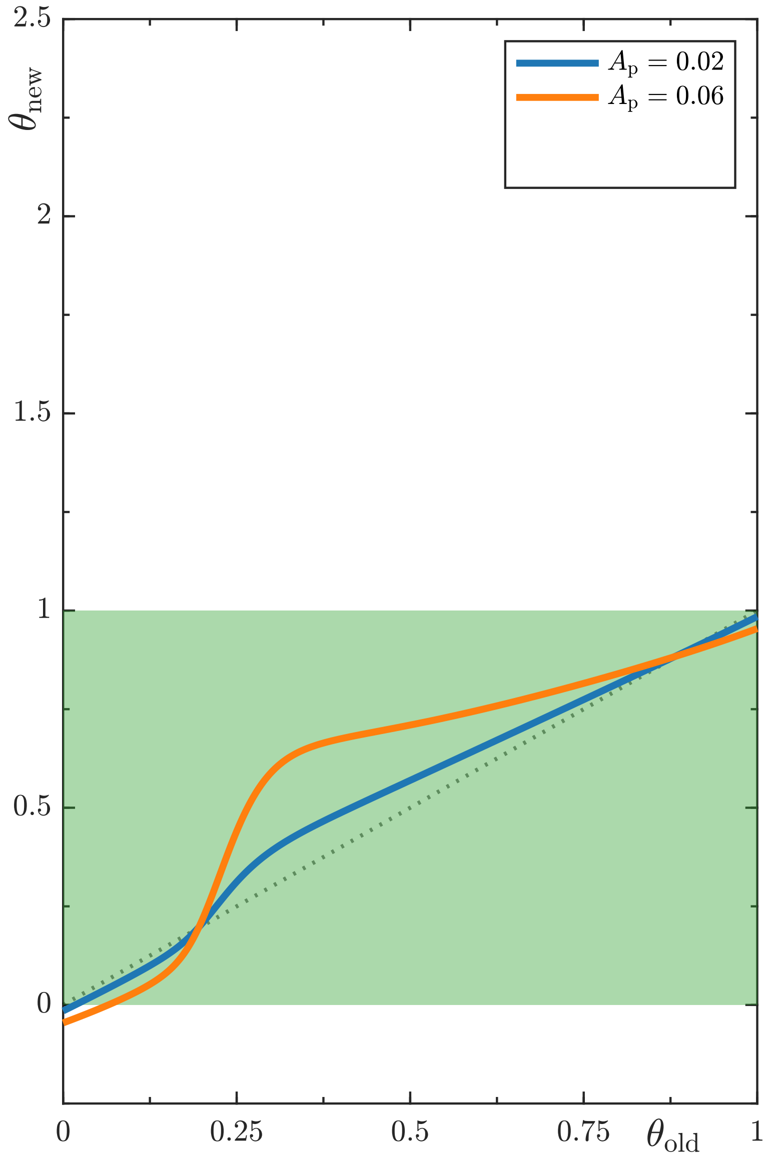


PHASE-TRANSITION CURVES (PTC)



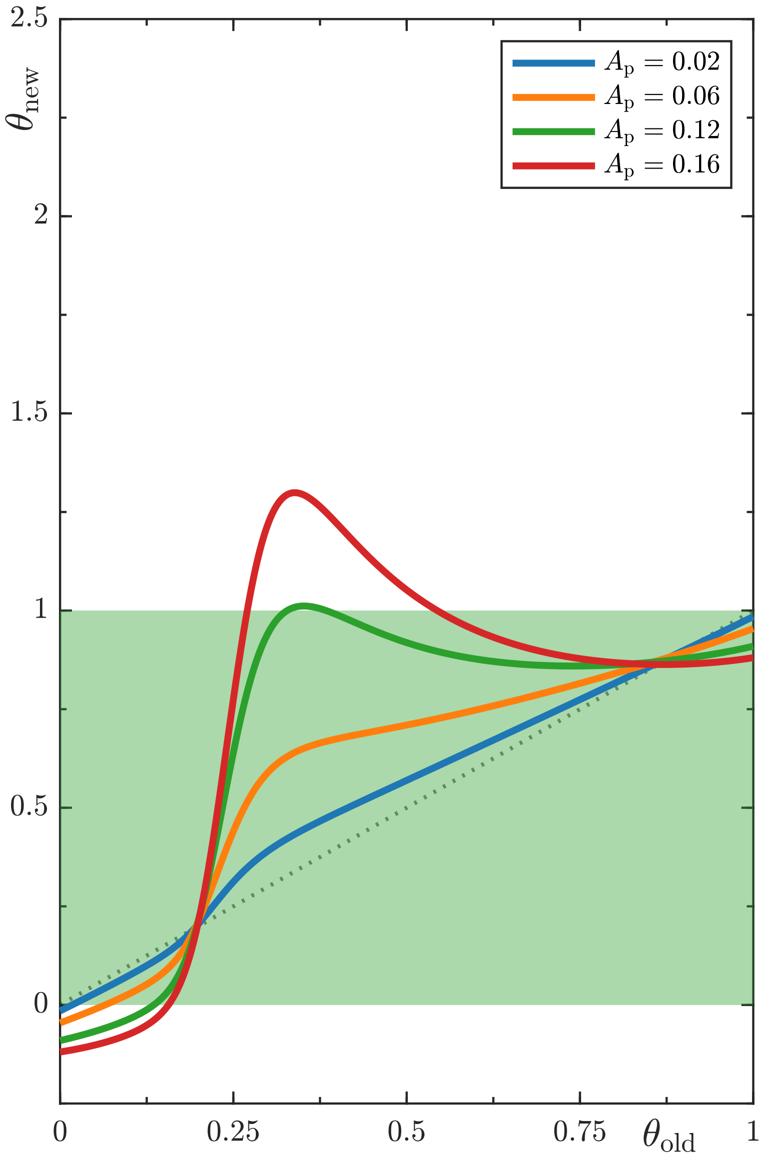
- Positive-G perturbations
  - $d_{\mathrm{p}} = (0, 0, 1)$
- Fundamental domain (green)
  - Represents full range of phases in the periodic orbit
- Weak perturbations "reset" to the same phase
  - $\theta_{\mathrm{p}} \approx \theta_{\mathrm{p}}$
- Stronger perturbations = "difference"

PHASE-TRANSITION CURVES (PTC)



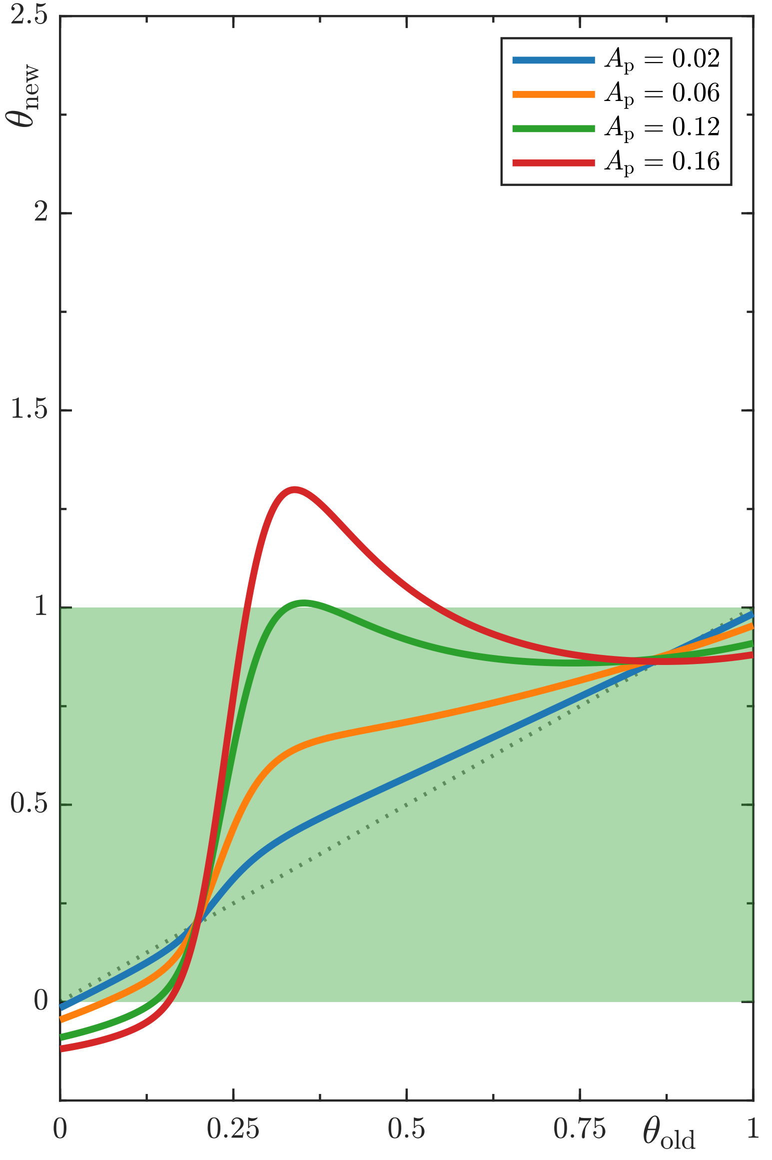
- Positive-G perturbations
  - $d_p = (0, 0, 1)$
- Fundamental domain (green)
  - Represents full range of phases in the periodic orbit
- Weak perturbations “reset” to the same phase
  - $\theta_p \approx \theta_p$
- Stronger perturbations = ”difference”

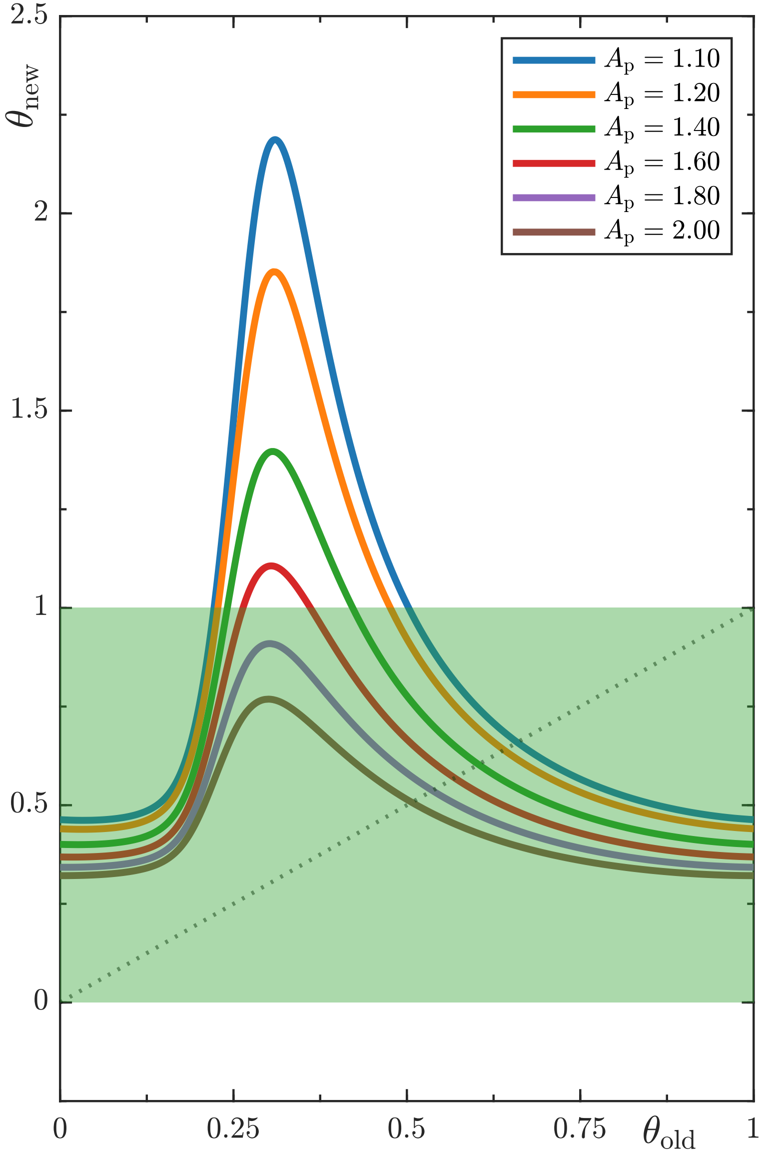
PHASE-TRANSITION CURVES (PTC)



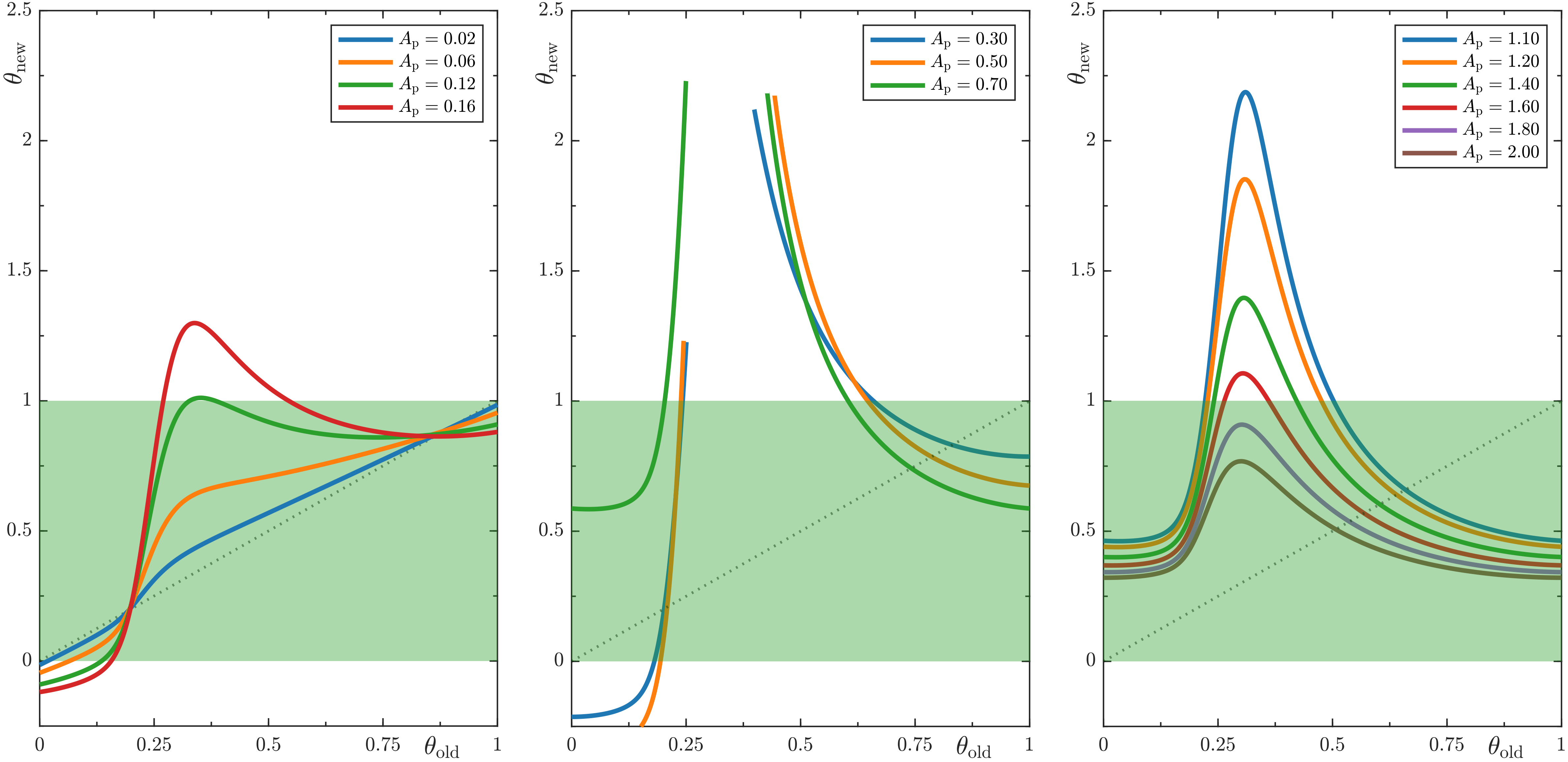
- Positive-G perturbations
  - $d_p = (0, 0, 1)$
- Fundamental domain (green)
  - Represents full range of phases in the periodic orbit
- Weak perturbations “reset” to the same phase
  - $\theta_p \approx \theta_p$
- Stronger perturbations = ”difference”

PHASE-TRANSITION CURVES (PTC)

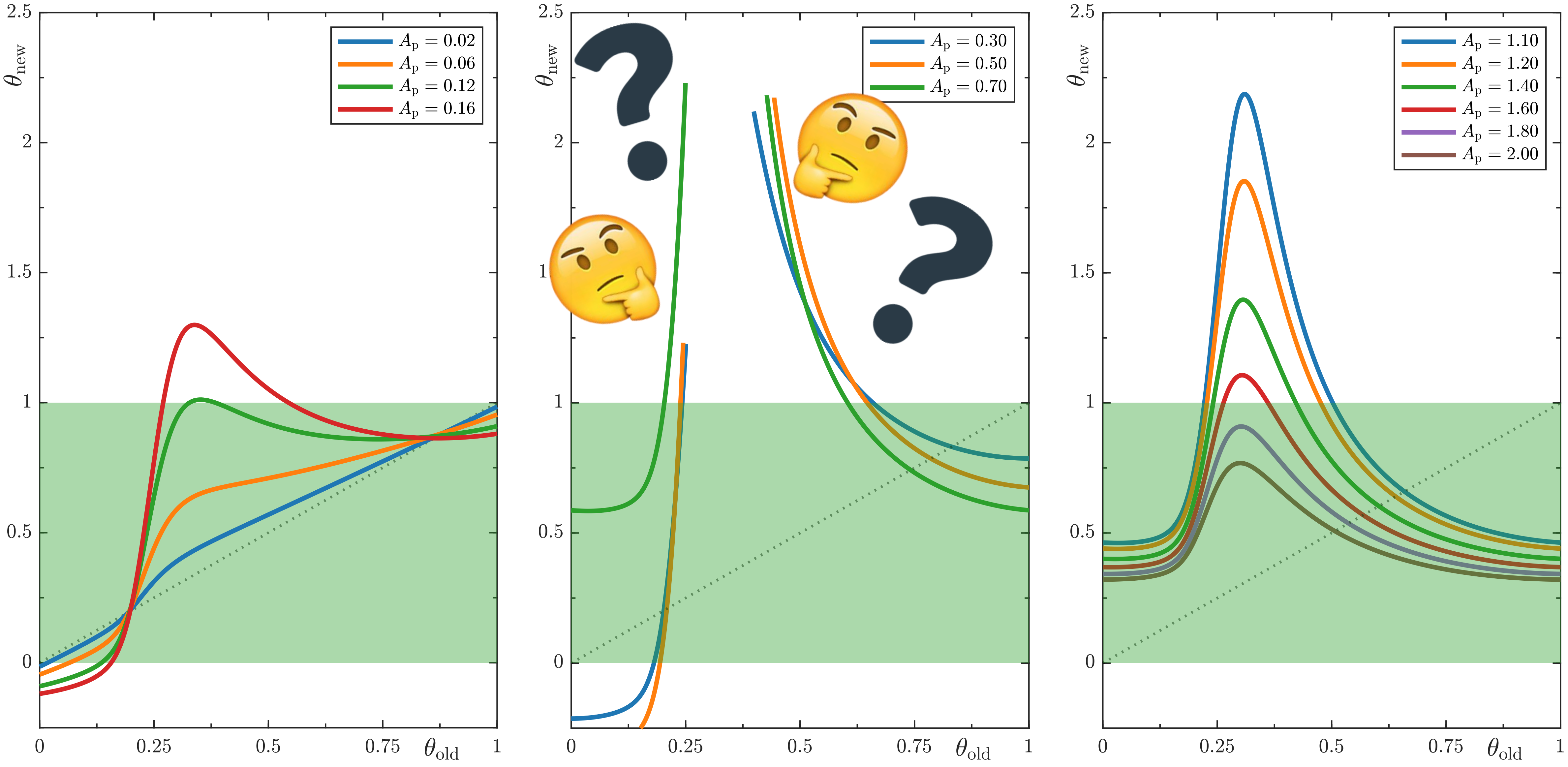




PHASE-TRANSITION CURVES (PTC)

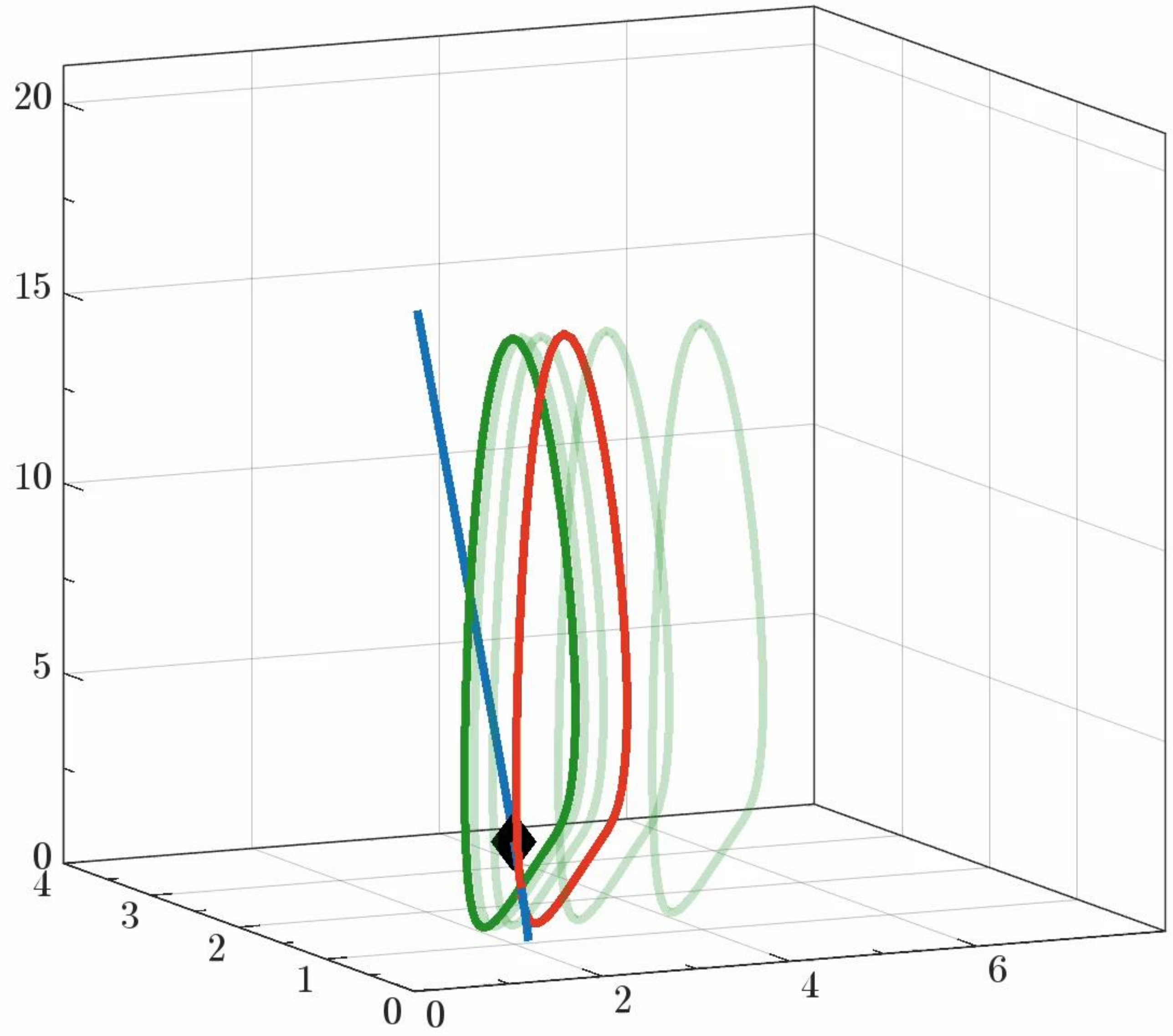


PHASE-TRANSITION CURVES (PTC)

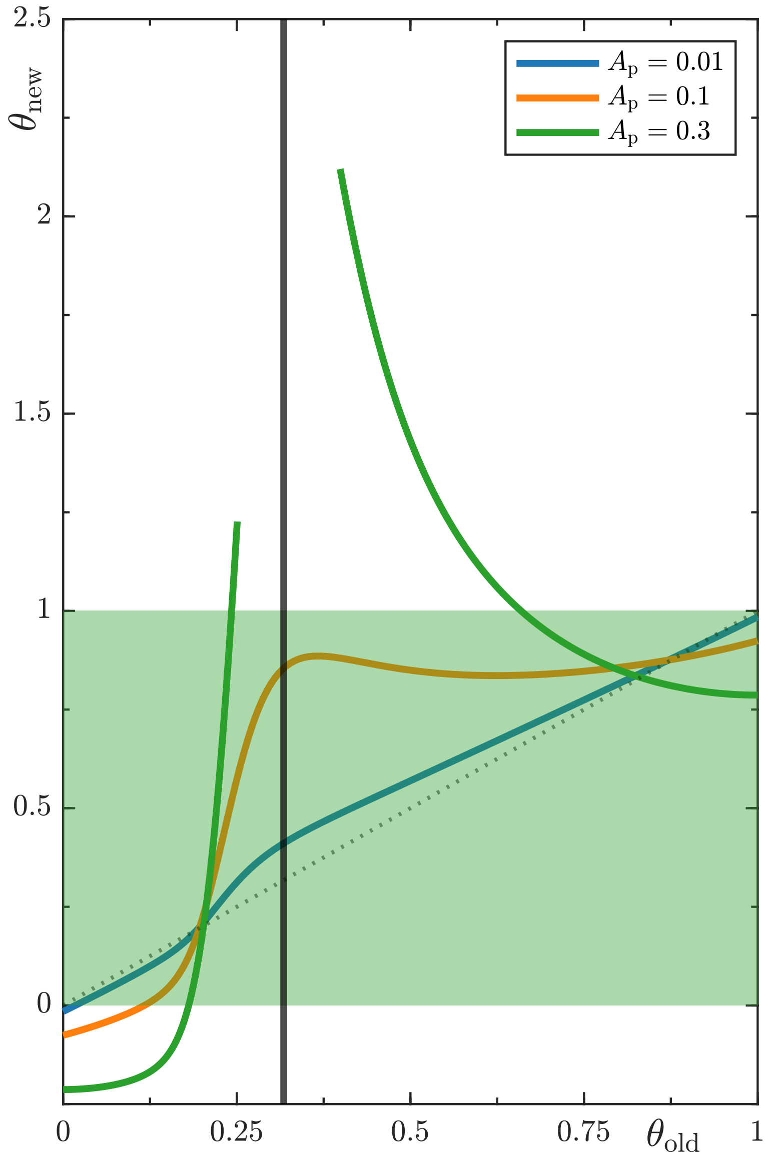


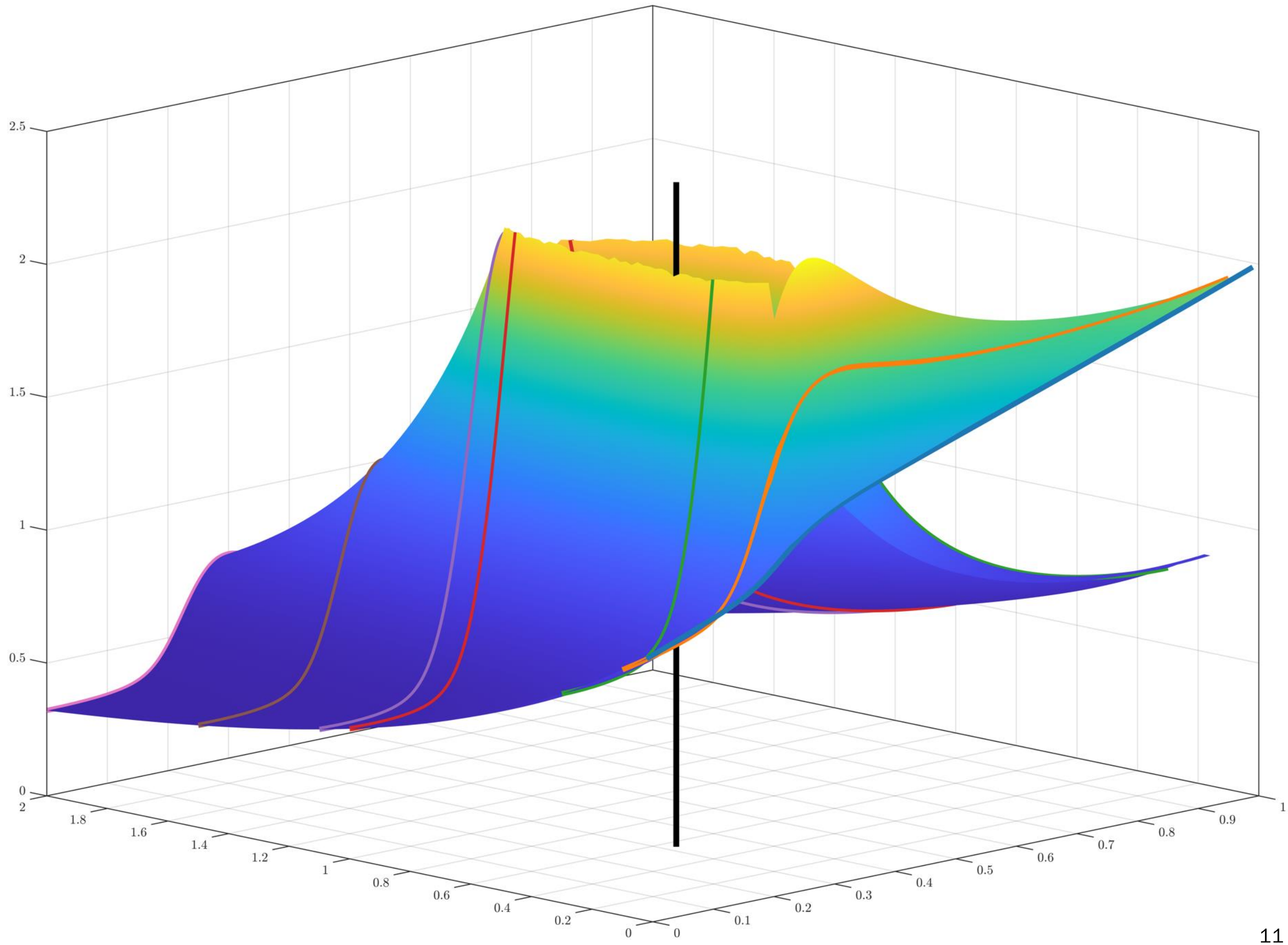
INTERSECTION WITH THE STABLE MANIFOLD

- Stable manifold of q intersects orbit $W^s(q)$
  - Initial point on stable manifold evolves towards q instead of “resetting”.
- Occurs at $\theta_{\mathrm{old}} \approx 0.35$ for $A_p \approx 0.55$.
- Each point along orbit will have some perturbation pushing it into $W^s(q)$
  - Combination of $A_{\mathrm{p}}, d_{\mathrm{p}}$, and θ_{old}.
- Returned phase θ_{new} grows until undefined

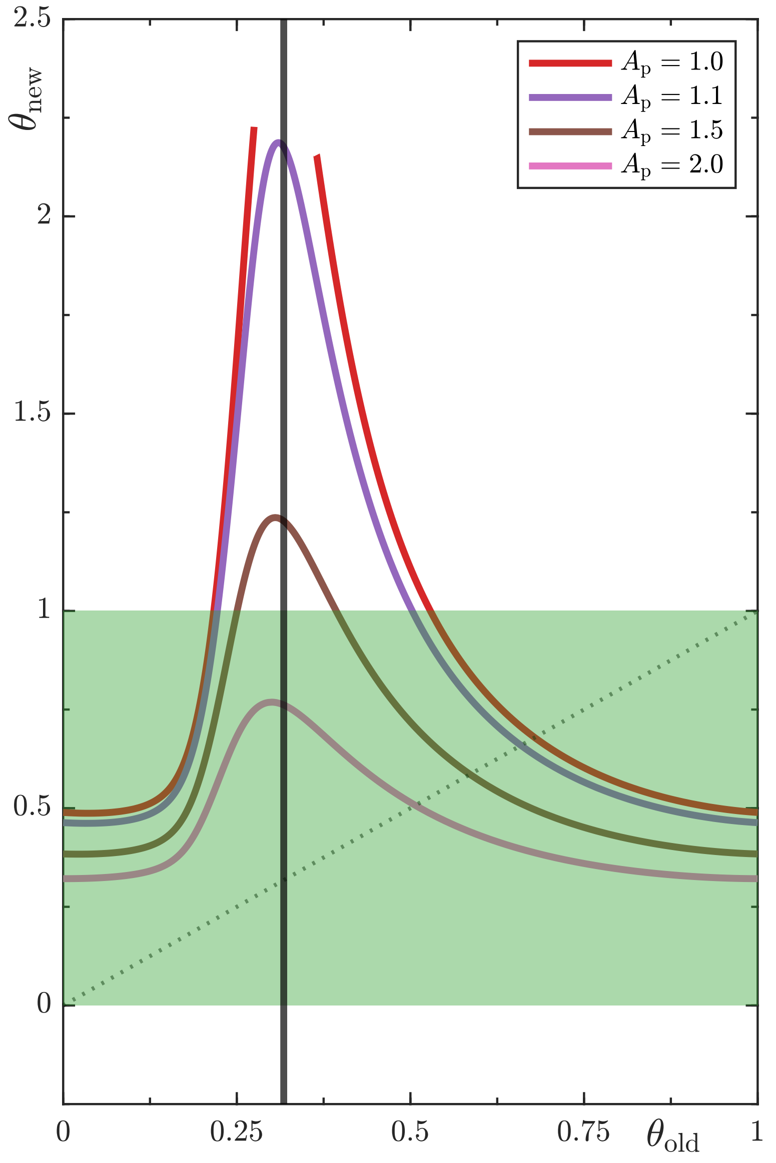


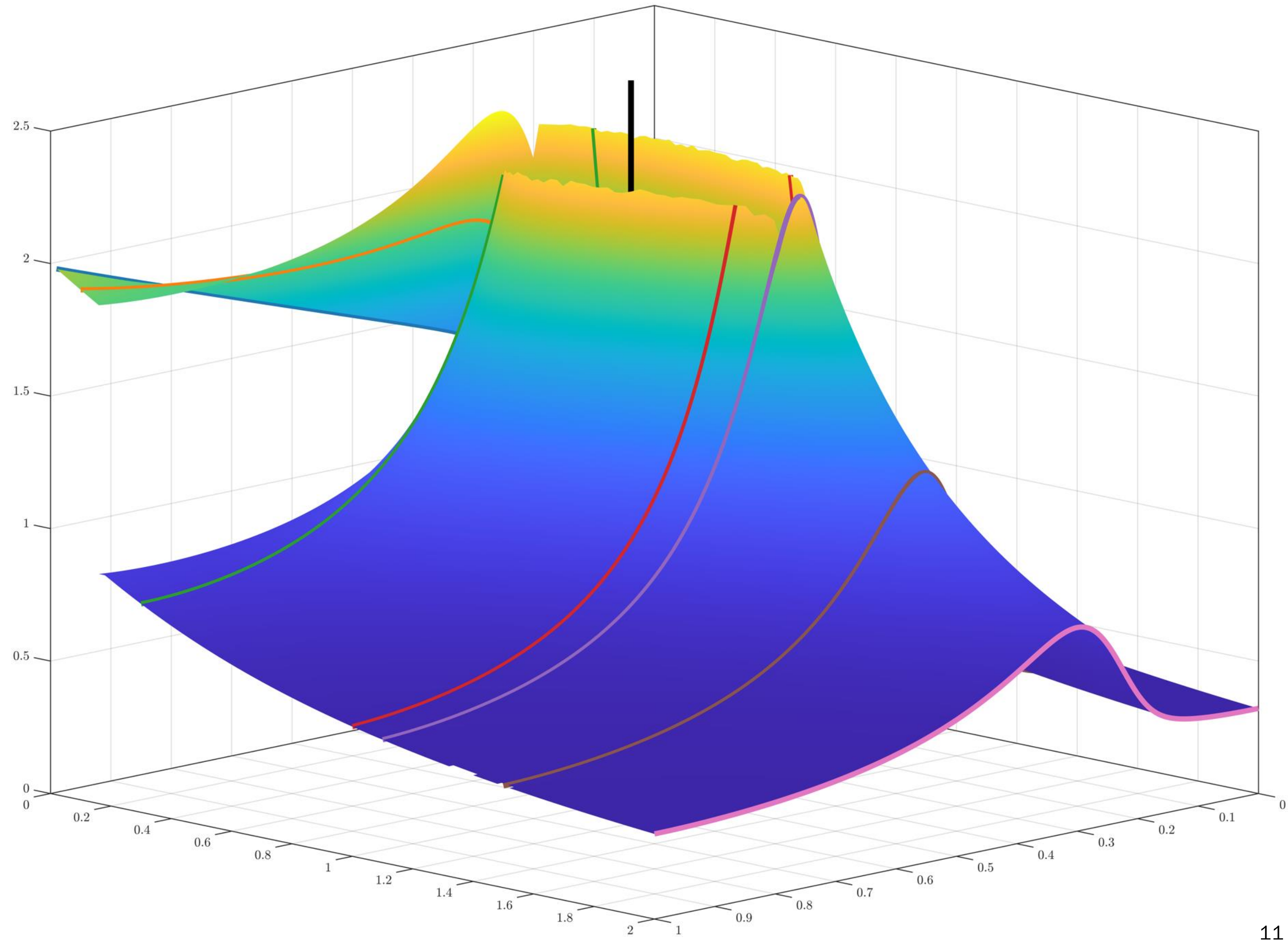
PTC SURFACE: *G*-PERTURBATION



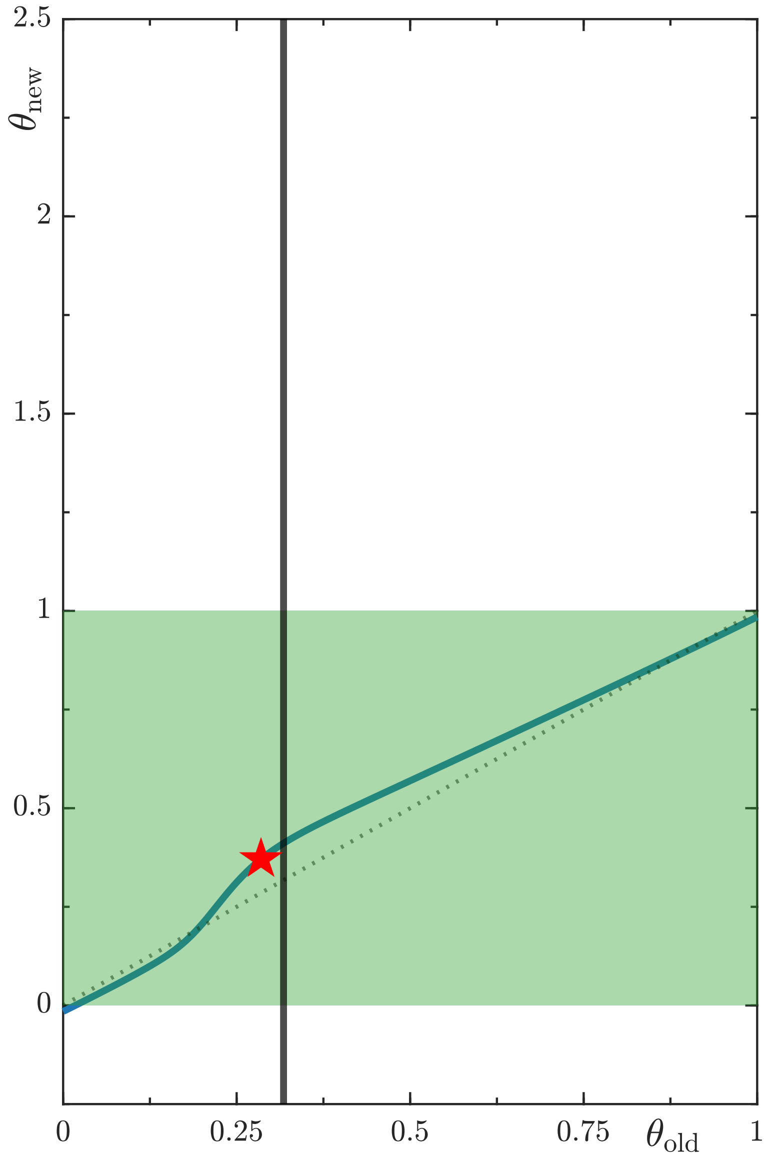


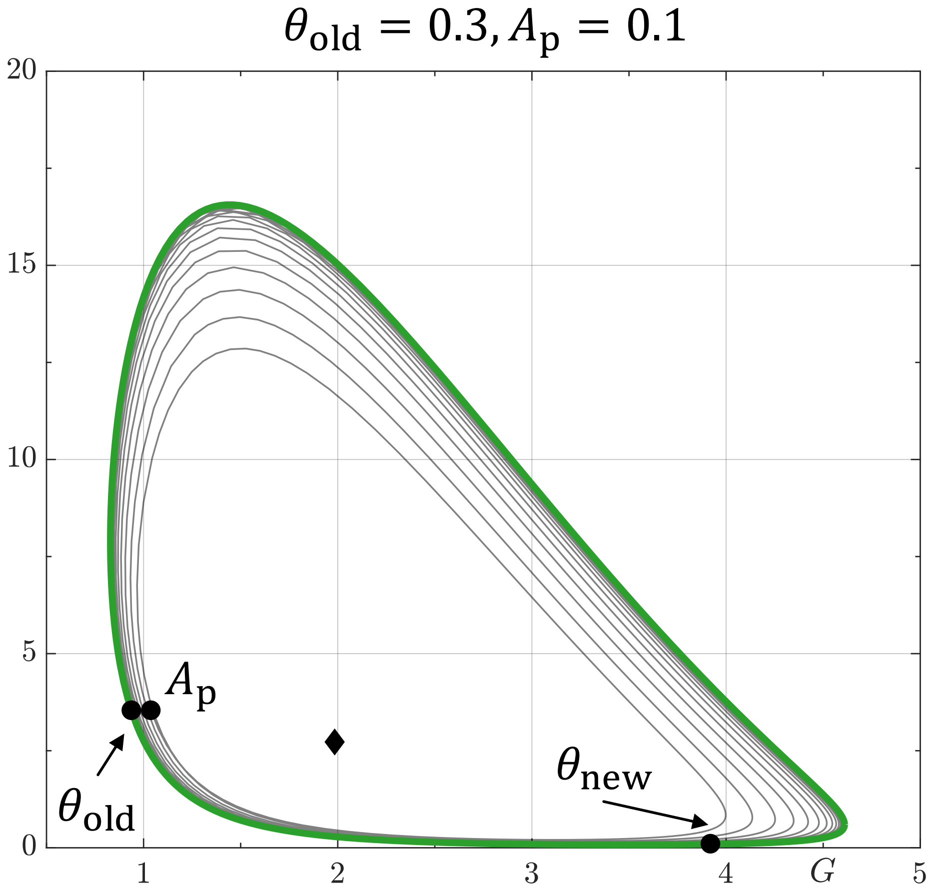
PTC SURFACE: *G*-PERTURBATION





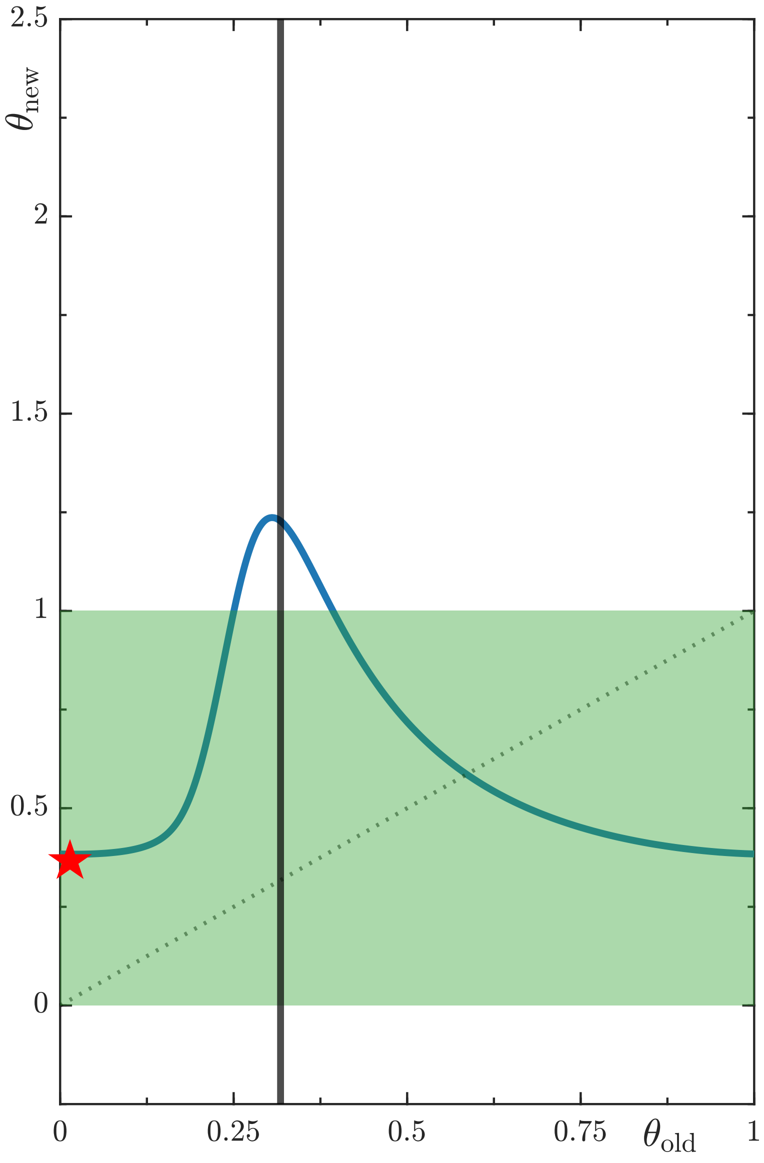
PTC "HOLE" – SLOW DYNAMICS

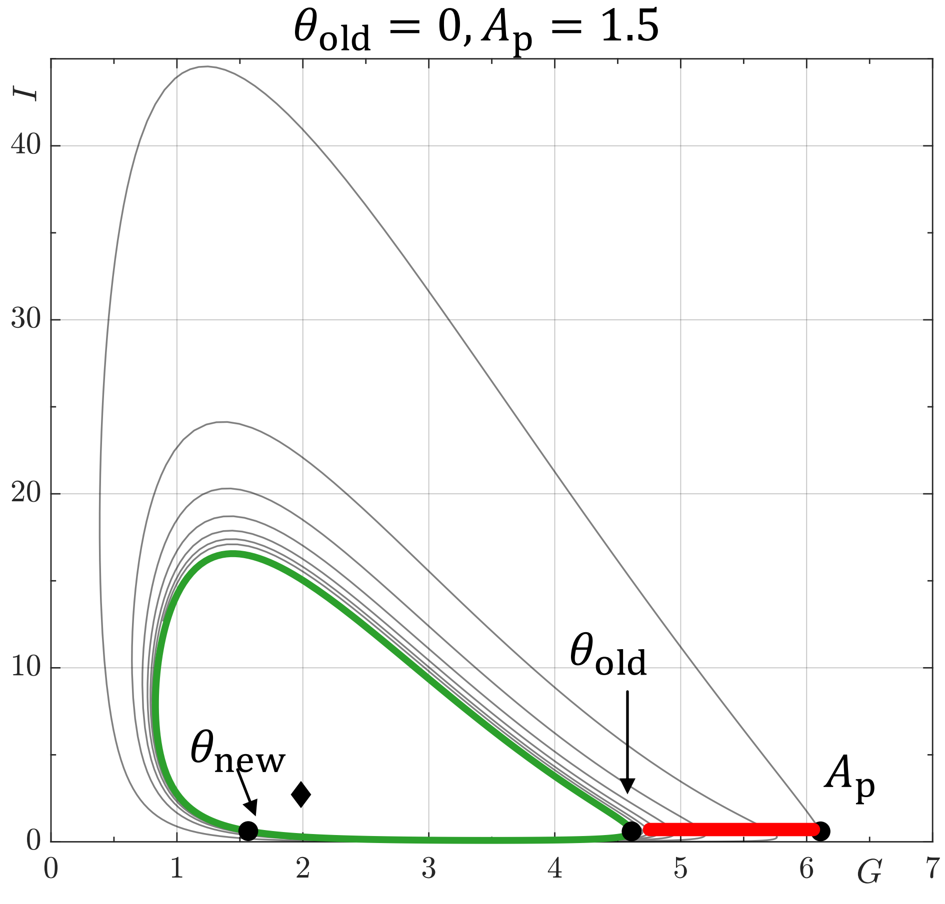




- Weak perturbations reset "quickly
- Strong perturbations "far away" from $W^s(q)$ reset "quickly"
- Perturbations close to $W^s(q)$ spiral for a long time
  - Longer than allowed computation ☹

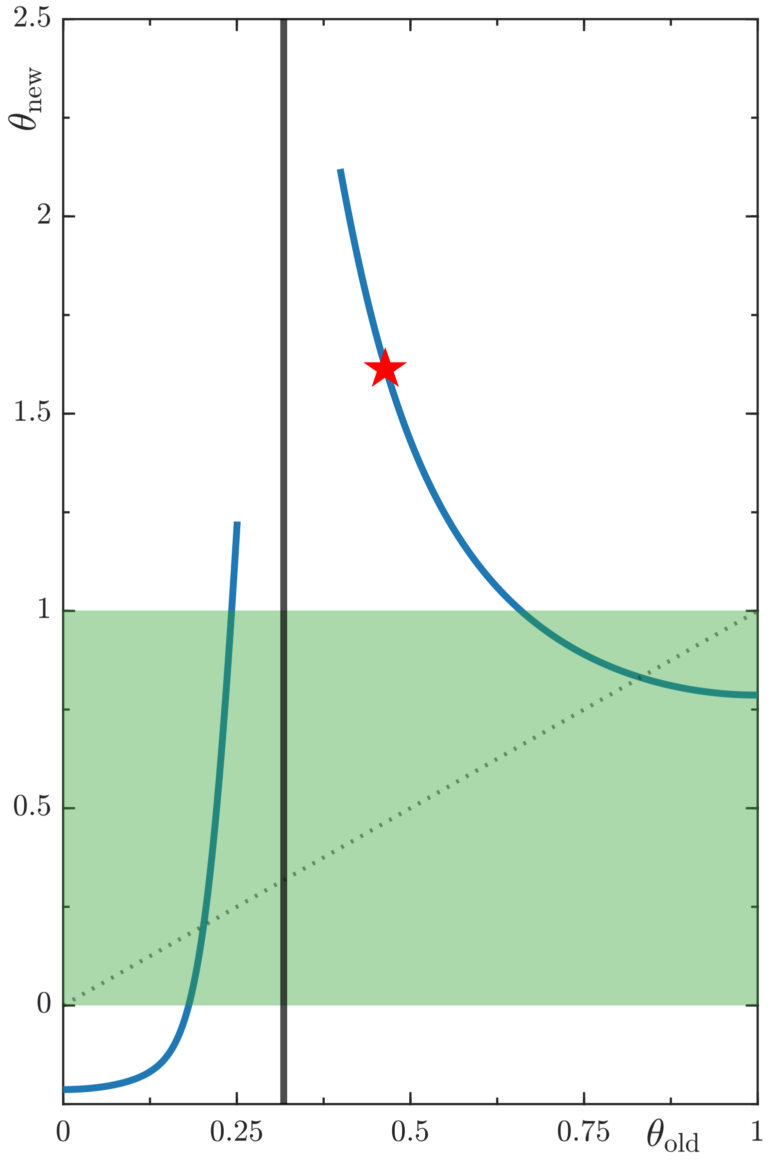
PTC "HOLE" – SLOW DYNAMICS

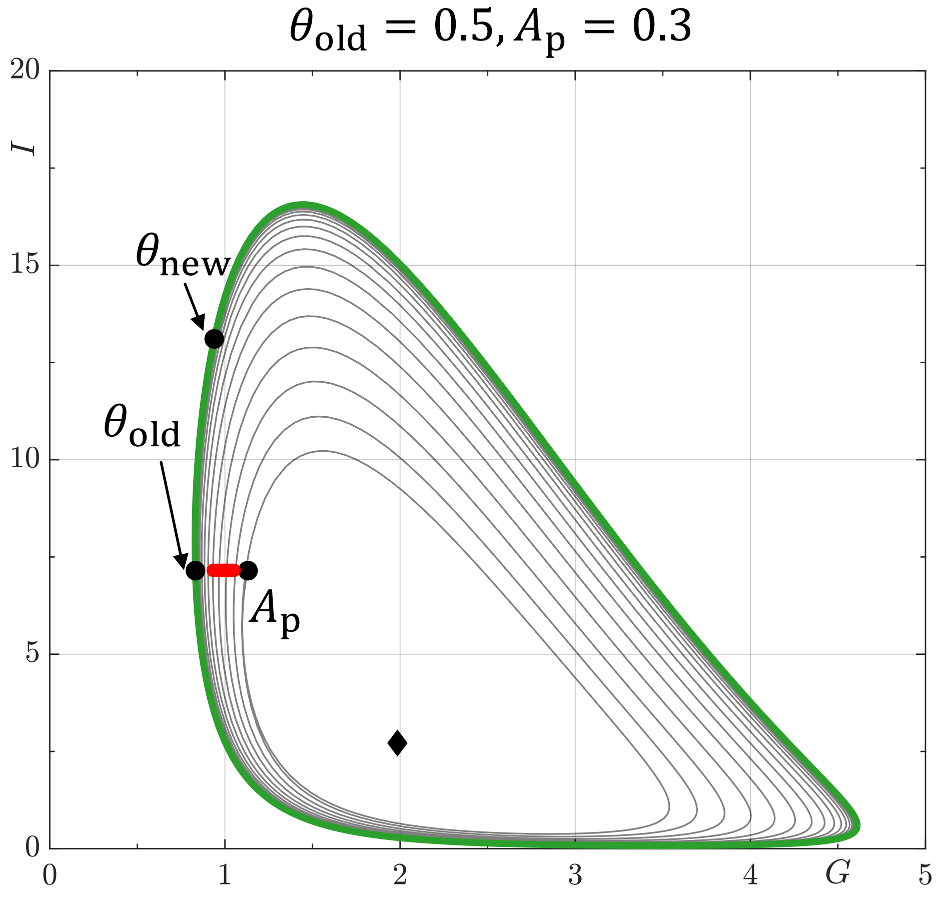




- Weak perturbations reset "quickly
- Strong perturbations "far away" from $W^s(q)$ reset "quickly"
- Perturbations close to $W^s(q)$ spiral for a long time
    - Longer than allowed computation ☹

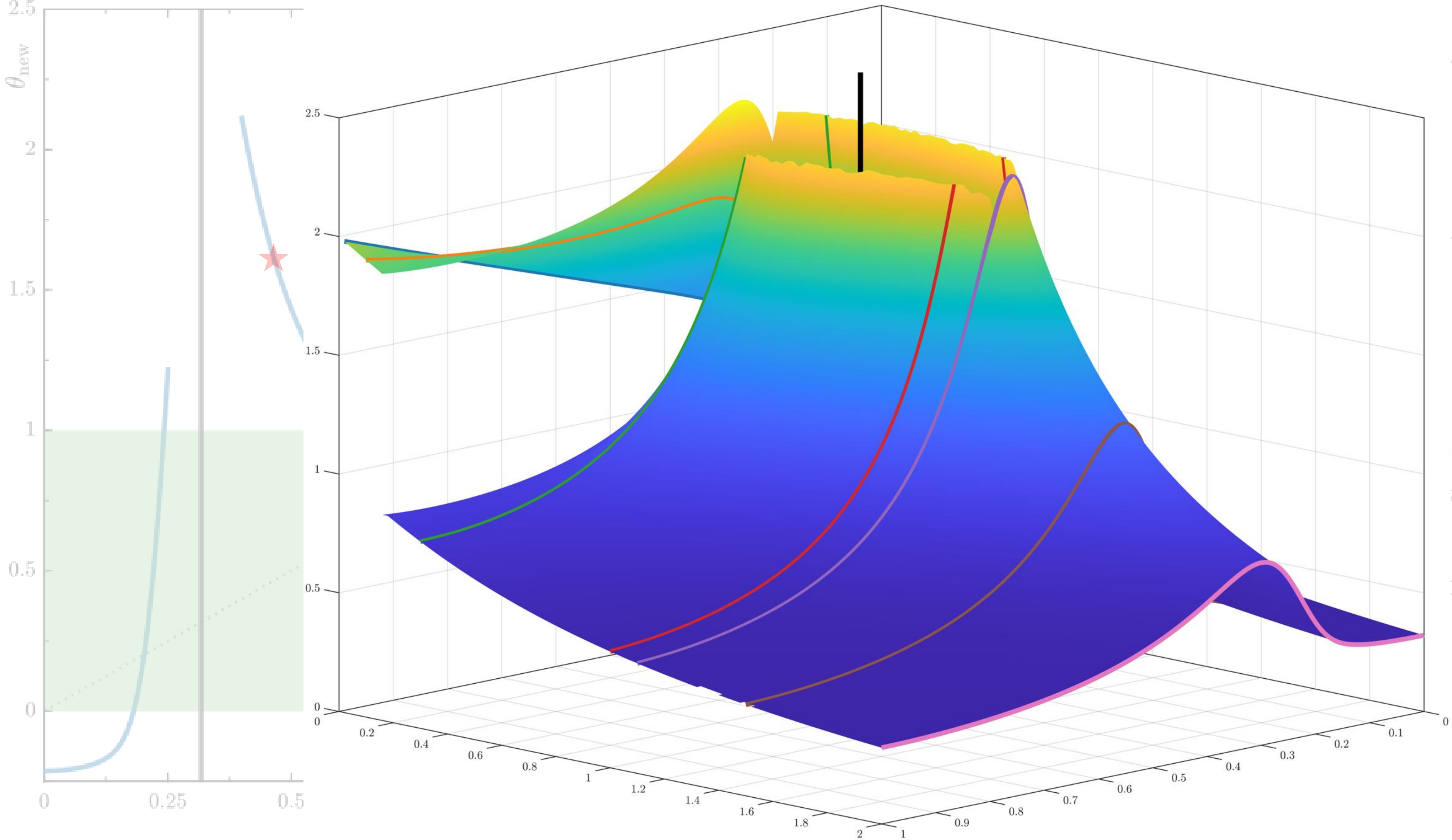
PTC "HOLE" – SLOW DYNAMICS





- Weak perturbations reset "quickly
- Strong perturbations "far away" from $W^s(q)$ reset "quickly"
- Perturbations close to $W^s(q)$ spiral for a long time
  - Longer than allowed computation ☹

PTC "HOLE" – SLOW DYNAMICS



urbations reset

urbations "far
$W^s(q)$ reset

ns close to
al for a long time

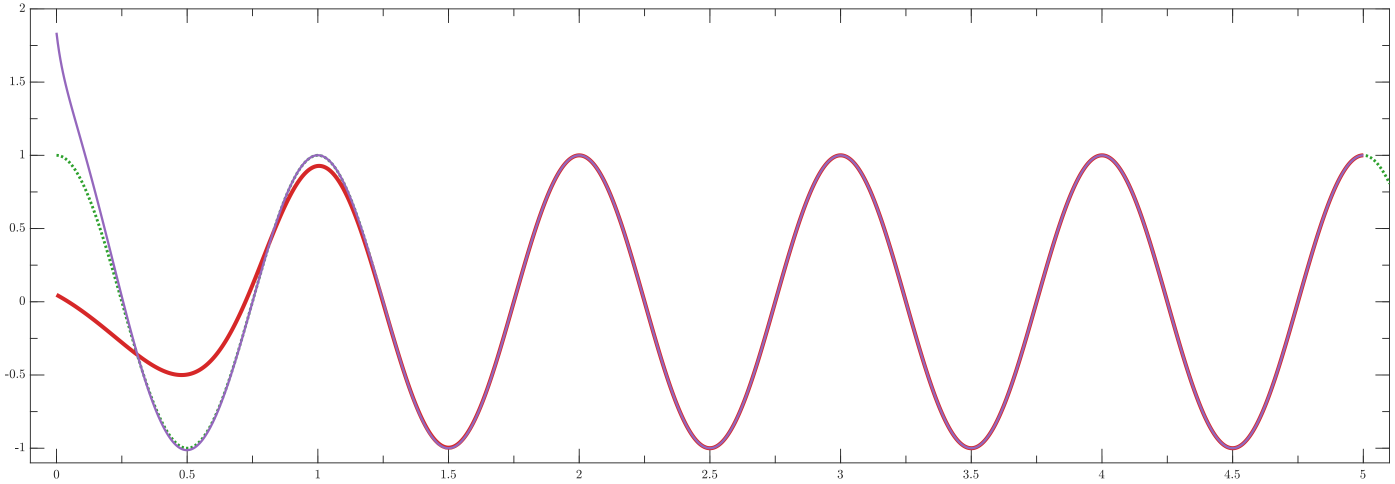
an allowed
tion ☹

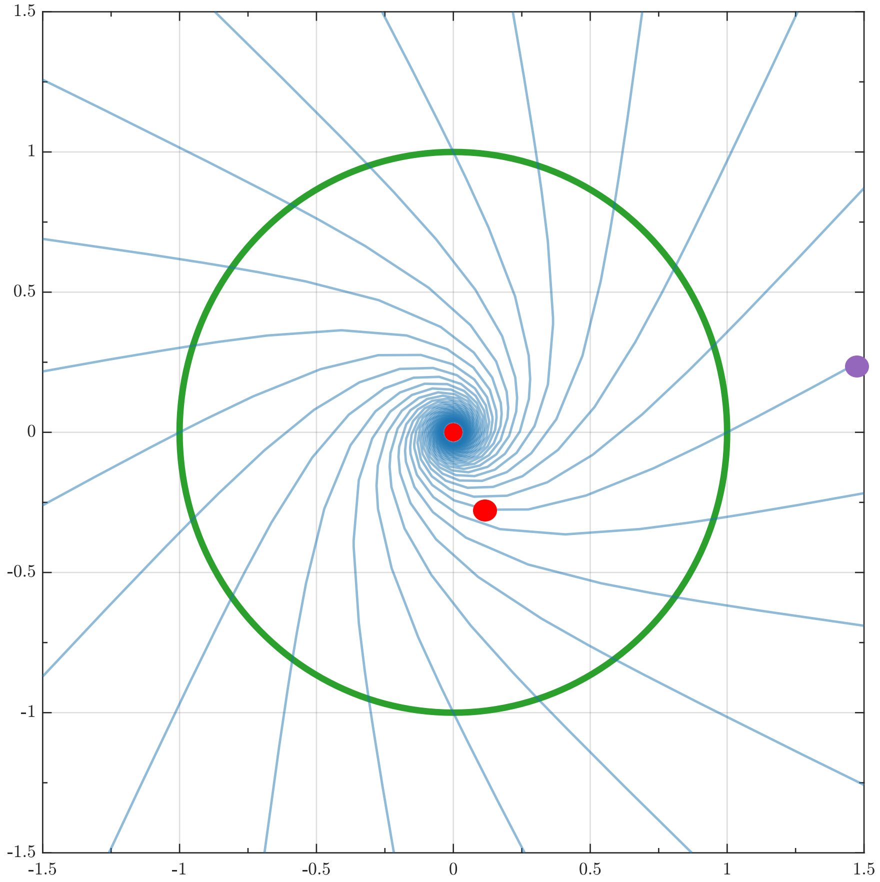
ISOCHRONS

- Isochrons are the set of all points which reset to the same phase
  - $\theta_{\text{old}} = \theta_{\text{new}}$

- Isochron associated with each point along the periodic orbit

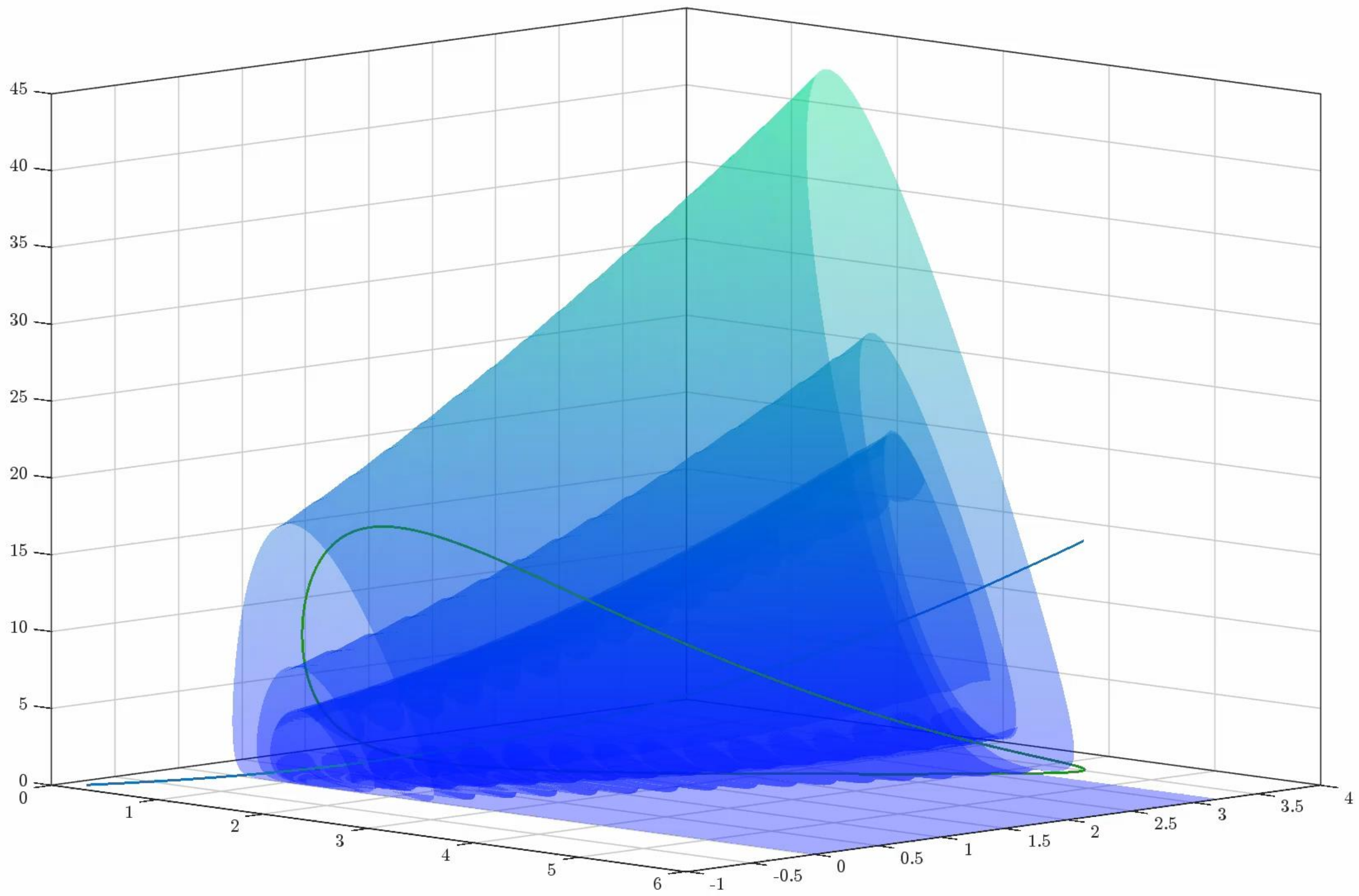
- All points in phase space have a unique phase depending on the isochron they lie on





TWO-DIMENSIONAL ISOCHRON

- Yamada model is a three-dimensional system
  - Each isochron is then a two-dimensional object
- "Carpet roll" around the stable manifold $W^s(q)$
- This isochron is for the head point γ_0
  - The "first" point of the periodic orbit
- As with a 2D model, there are isochrons for each point along the periodic orbit



CONCLUSIONS

- Phase-Resetting is a powerful tool in studying the response of periodic solutions to induced perturbations
- Discontinuities in PTC when perturbation approaches stable manifold of spiral source
- Can technically consider perturbation in any “direction”.