PHASE-RESETTING IN THE YAMADA MODEL OF A *Q*-SWITCHING LASER

PHASE RESPONSE FROM AN INDUCED PERTURBATION

JACOB NGAHA, NEIL G. R. BRODERICK, AND BERND KRAUSKOPF

DODD-WALLS CENTRE SYMPOSIUM

18TH NOEMA, 2024

TE WHAI AO
DODD-WALLS CENTRE
for Photonic and Quantum Technologies

UNIVERSITY OF AUCKLAND
Waipapa Taumata Rau
NEW ZEALAND

STABLE Q-SWITCHING LASERS

- Optical frequency combs and optical clocks need stability
- How do they return to equilibrium when perturbed?
- Q-switching lasers can be optical analogues to neurons
  - Optical neural networks

Journal of the Optical Society of America B
OPTICAL PHYSICS

Excitability in an all-fiber laser with a saturable absorber section

ROBERT OTUPIRI,[1,*] BRUNO GARBIN,[2] NEIL G. R. BRODERICK,[1] AND BERND KRAUSKOPF[3,4]

All optical Q-switched laser based spiking neuron

Keshia Mekemeza-Ona, Baptiste Routier and Benoît Charbonnier*

Université Grenoble-Alpes, CEA, Leti, Grenoble, France

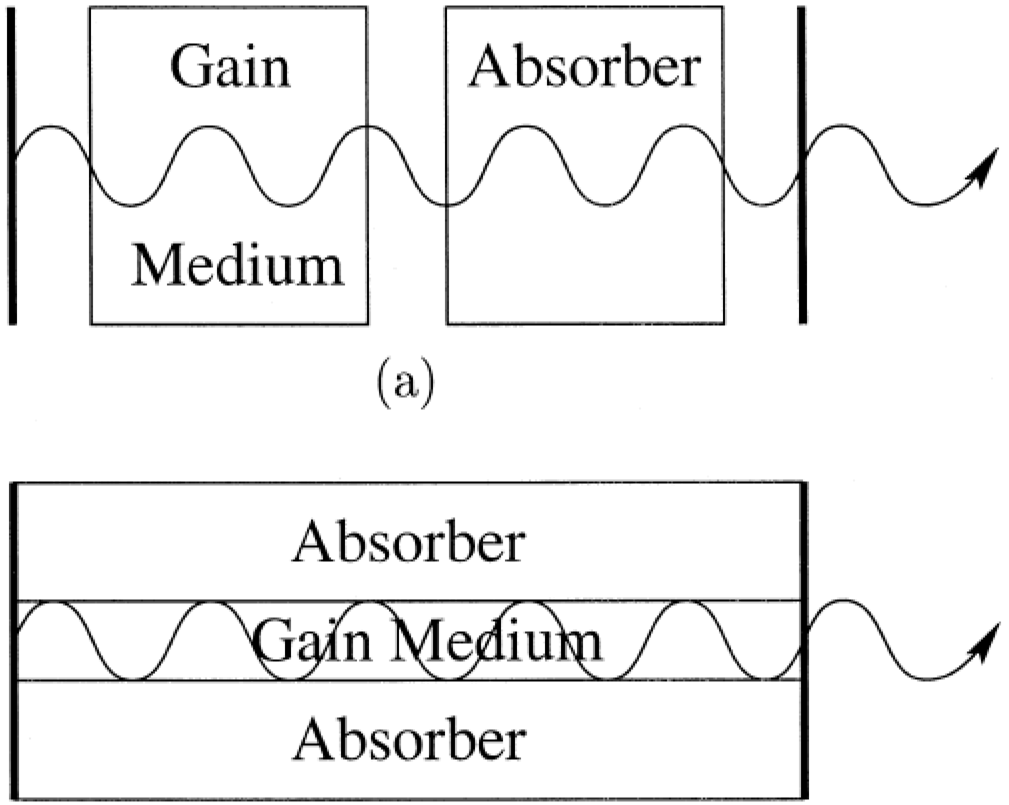
THE YAMADA MODEL

$$\dot{G} = \gamma\,(A - G - G\,I)$$
$$\dot{Q} = \gamma\,(B - Q - a\,Q\,I)$$
$$\dot{I} = (G - Q - 1)\,I$$

- G – Gain
- Q – Absorption
- I – Intensity

Parameters

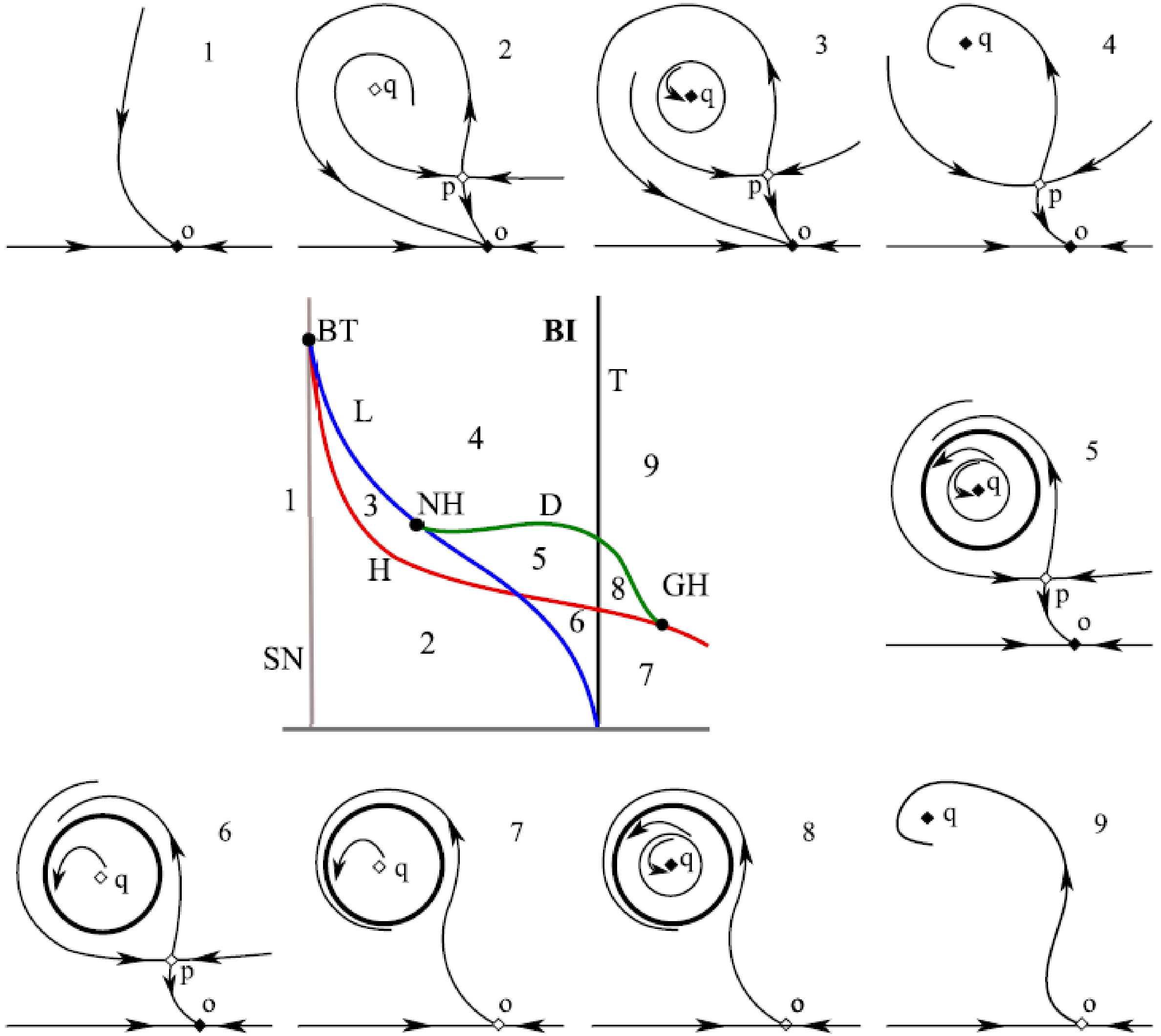
- γ – Photon loss rate
- A – Pump current to gain
- B – Absorption coefficient
- a – Relative absorption vs. gain



Taken from J. L. A. Dubbeldam and B. Krauskopf *"Self-pulsations of lasers with saturable absorber: Dynamics and bifurcations"*, Opt. Commun., **159** (4-6), 325 (1999).

THE YAMADA MODEL: BIFURCATION DIAGRAM

- Different dynamics split by bifurcations:
  - Hopf, homoclinic, saddle
- Objects in phase space
  - *o* – Stable equilibrium (‘off state’)
  - *p* – Saddle with two unstable and one stable eigenvalues
  - *q* – Spiral source
  - Attracting periodic orbit
  - Saddle periodic orbit



Taken from R. Otupiri, B. Krauskopf, N. G. R. Broderick *“The Yamada Model for a Self-Pulsing Laser: Bifurcation Structure for Non Identical Decay Times of Gain and Absorber”*, Int. J. Bifurc. Chaos Appl. Sci. Eng., **30** (14) (2020).

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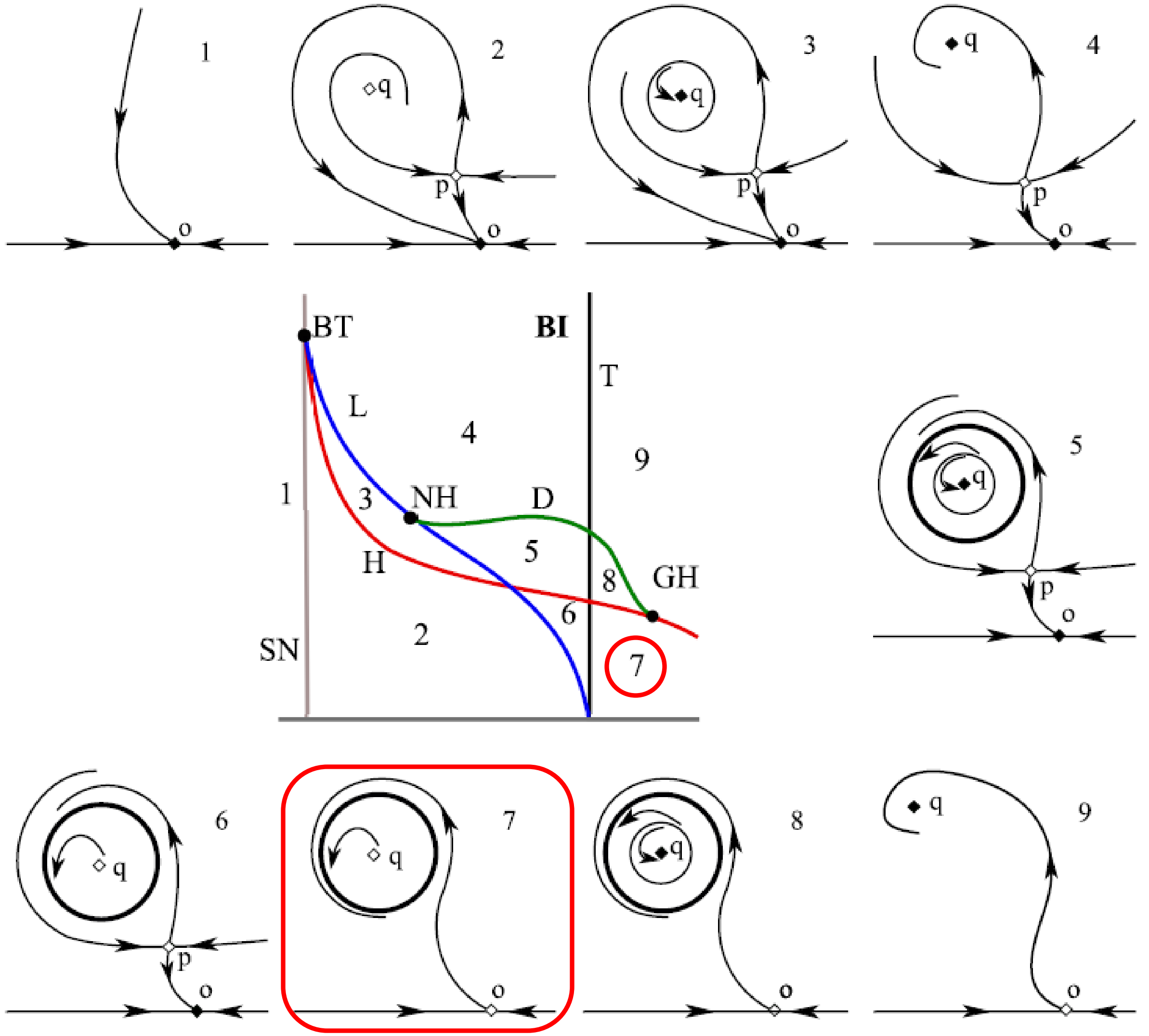

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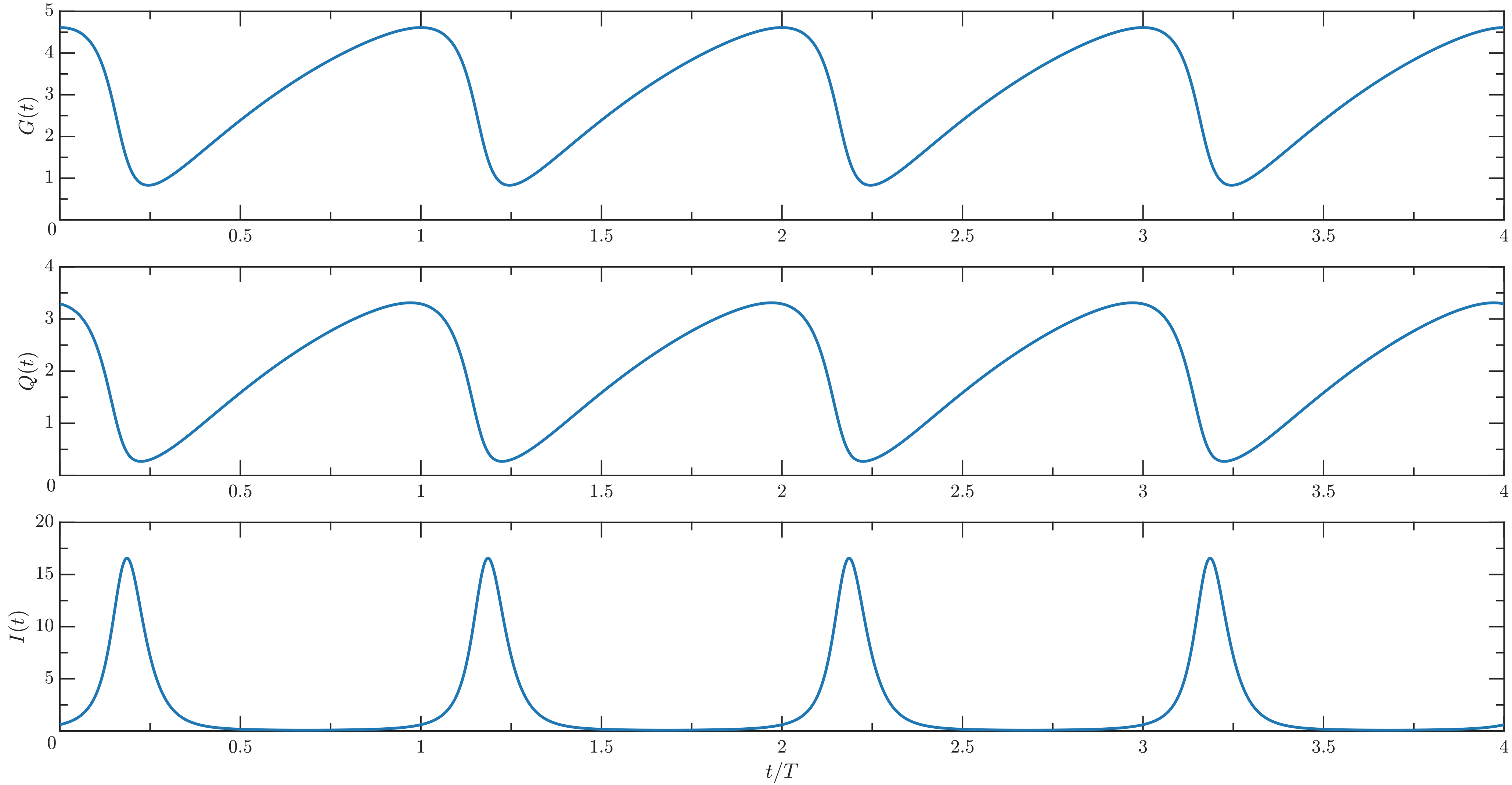
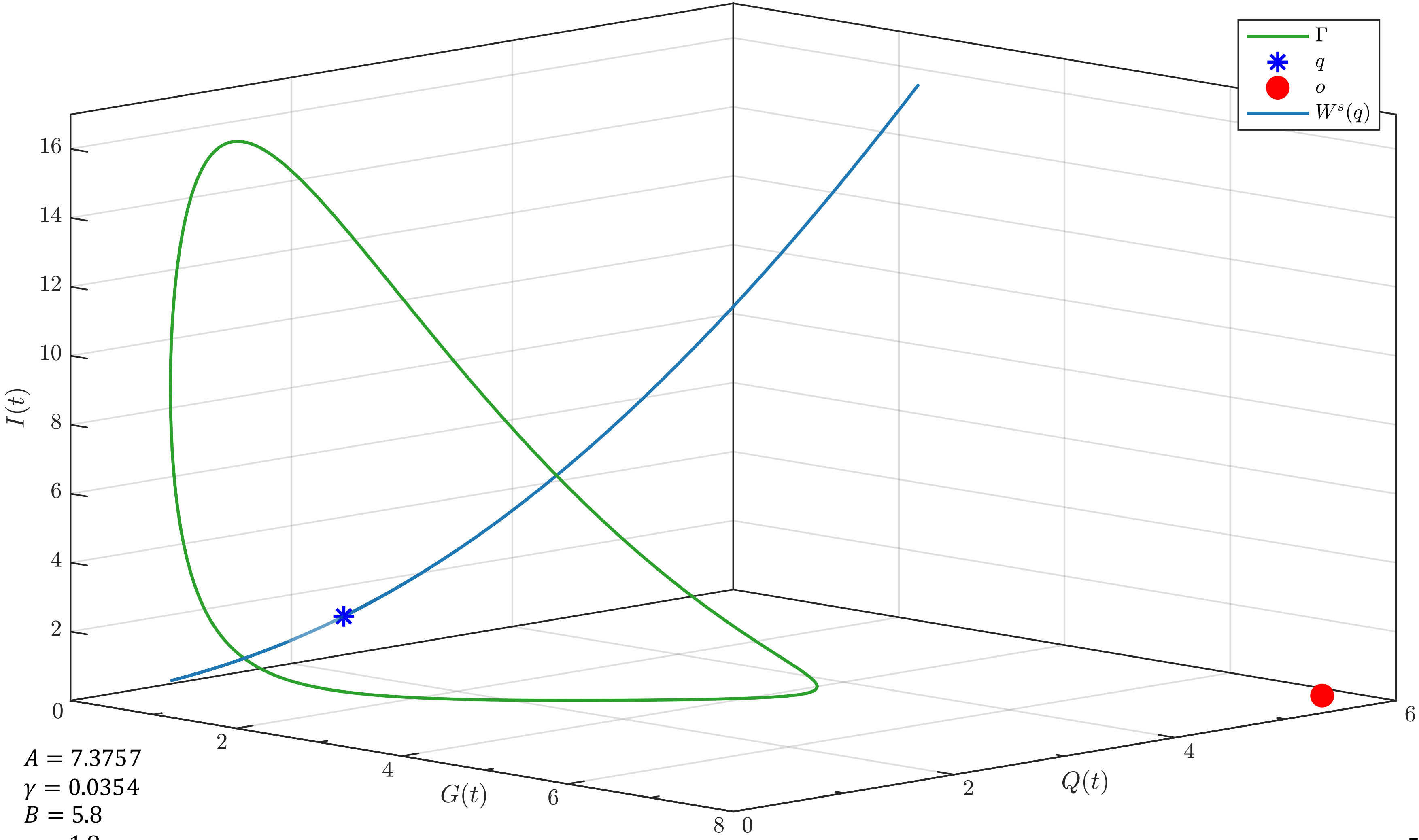
THE YAMADA MODEL: ATTRACTING PERIODIC ORBIT



$A = 7.3757$
$\gamma = 0.0354$
$B = 5.8$
$a = 1.8$

THE YAMADA MODEL: ATTRACTING PERIODIC ORBIT

- Attracting periodic orbit (green)
- “Off” state (red circle)
- Saddle (blue star)
  - 1-D stable manifold (blue)



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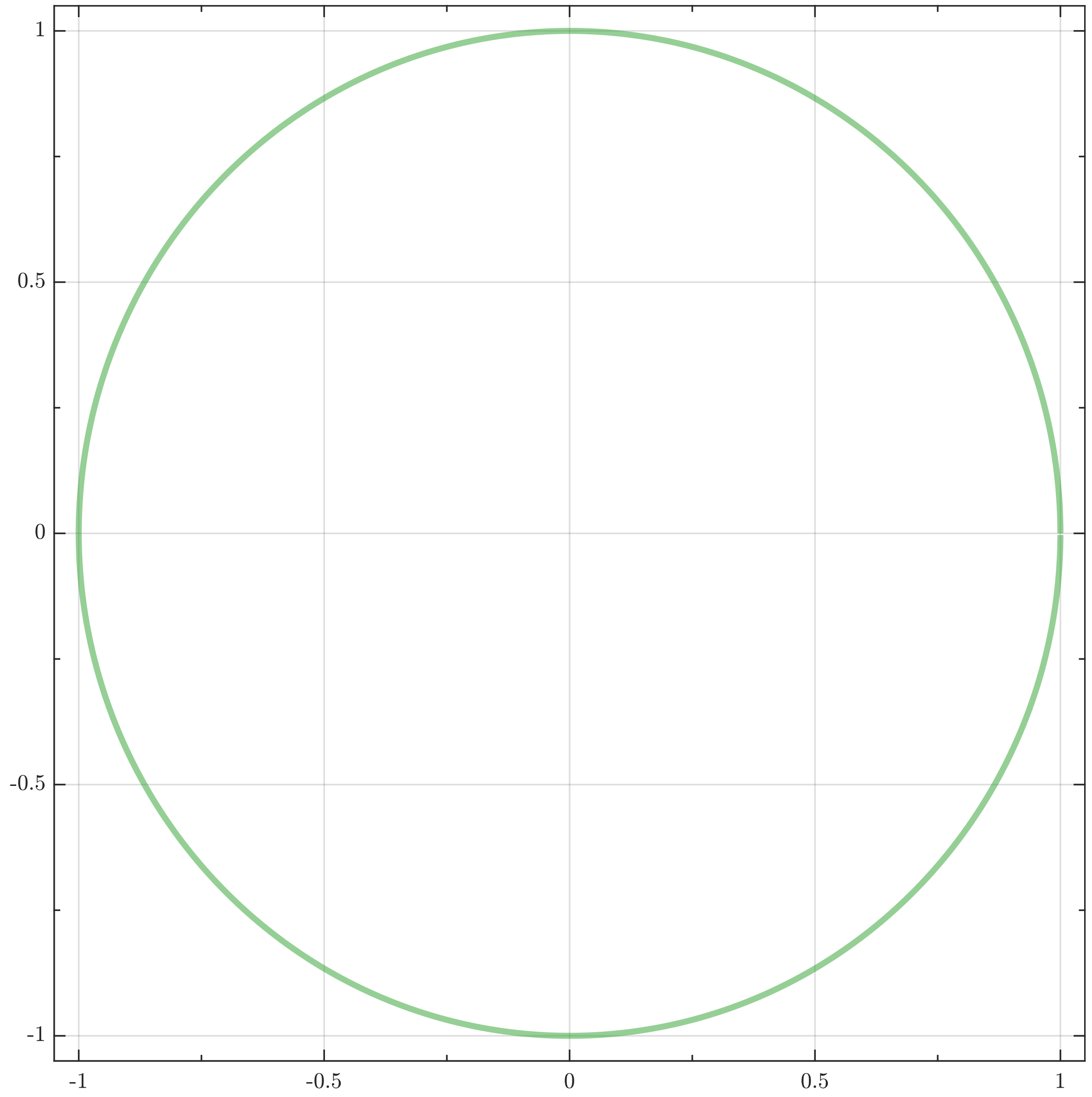
PHASE-RESETTING

- Induced perturbation
  - A_p – amplitude
  - $d_p = (\cos\theta_p, \sin\theta_p)$ – direction
  - θ_{old} – phase perturbation is applied
- When does the perturbed segment return?
  - θ_{new} – phase perturbation returns
- Boundary value problem (BVP)
  - Numerical continuation in AUTO and COCO



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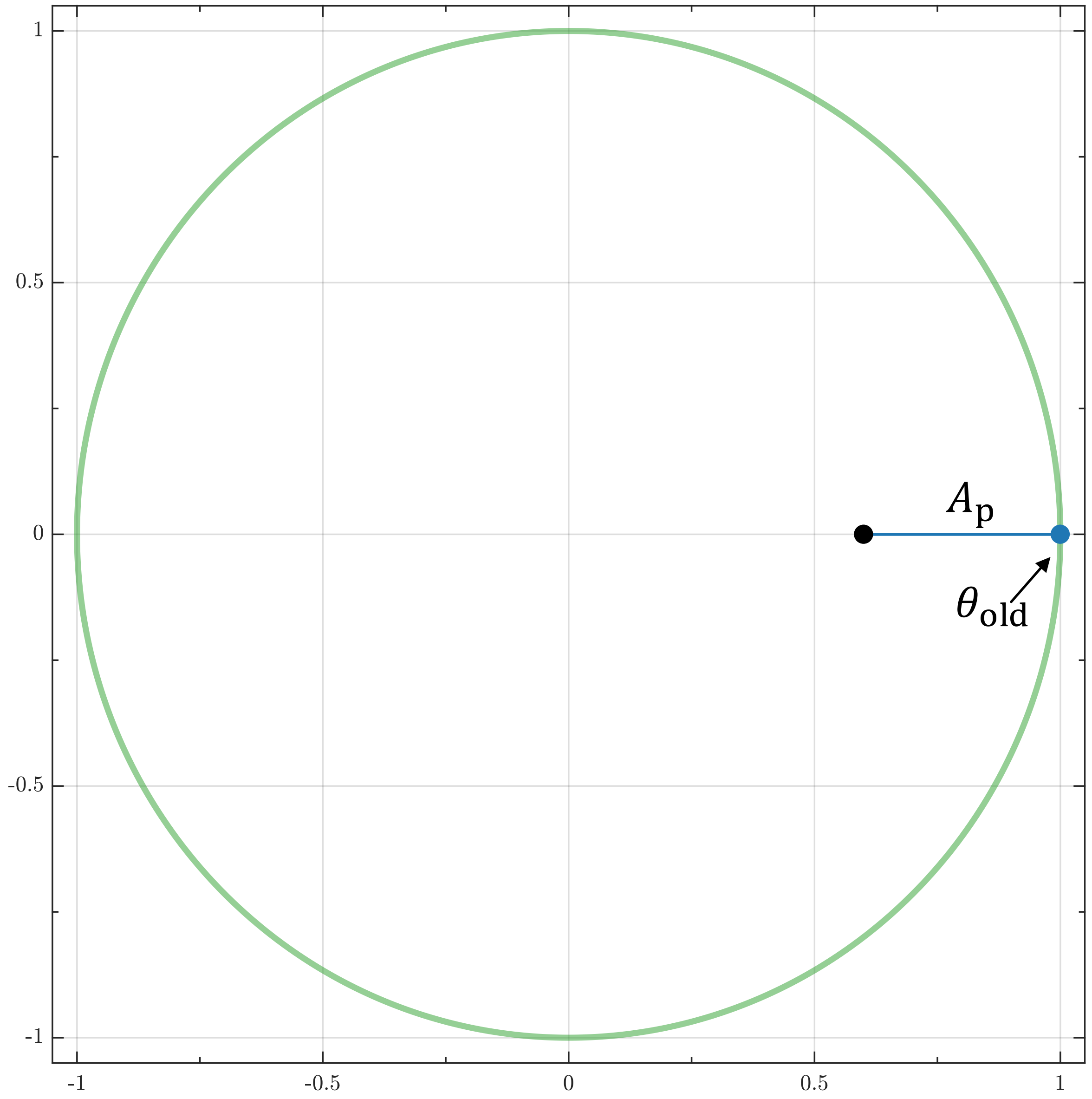

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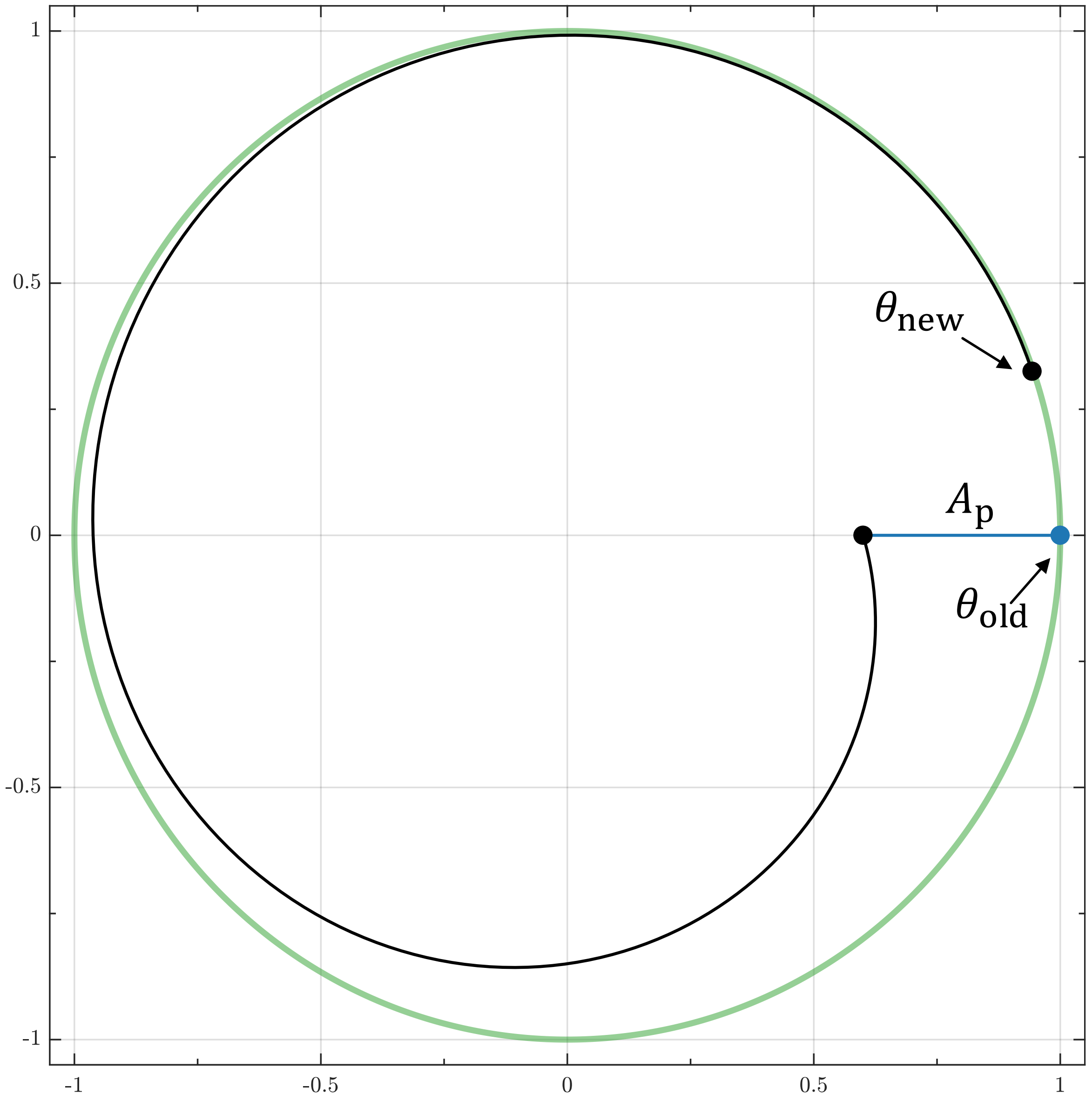

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A Continuation Approach to Computing Phase Resetting Curves

Peter Langfield[1,2], Bernd Krauskopf[3], and Hinke M. Osinga[3(✉)]

Phase response to arbitrary perturbations: Geometric insights and resetting surfaces

Kyoung H. Lee[1], Neil G. R. Broderick[2], Bernd Krauskopf[1] and Hinke M. Osinga[1]

SIAM J. Applied Dynamical Systems
Vol. 14, No. 3, pp. 1418–1453

Forward-Time and Backward-Time Isochrons and Their Interactions*

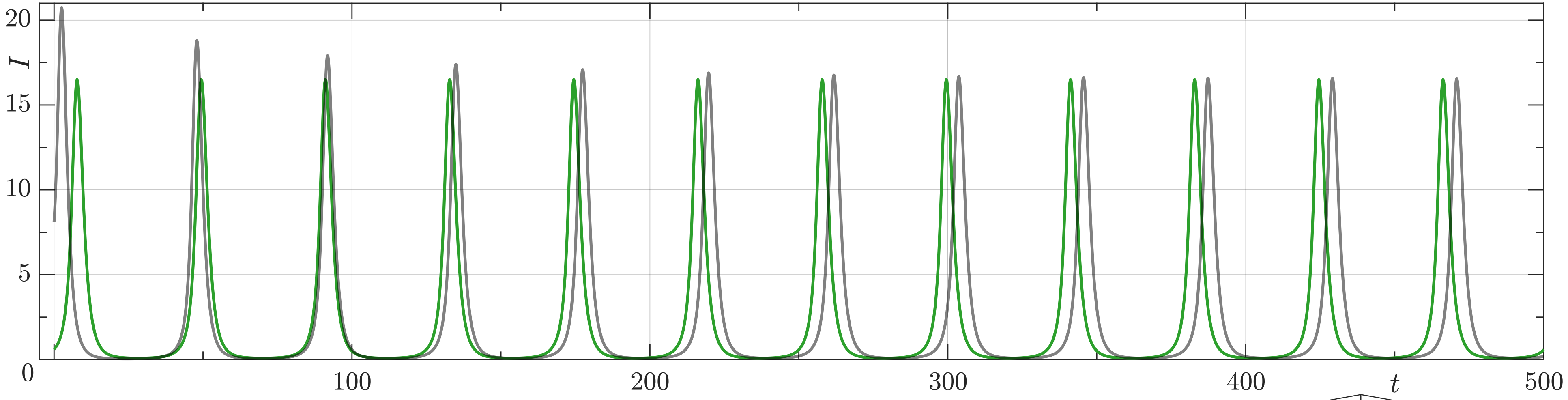
Peter Langfield†, Bernd Krauskopf†, and Hinke M. Osinga†

SIAM J. Applied Dynamical Systems
Vol. 9, No. 4, pp. 1201–1228

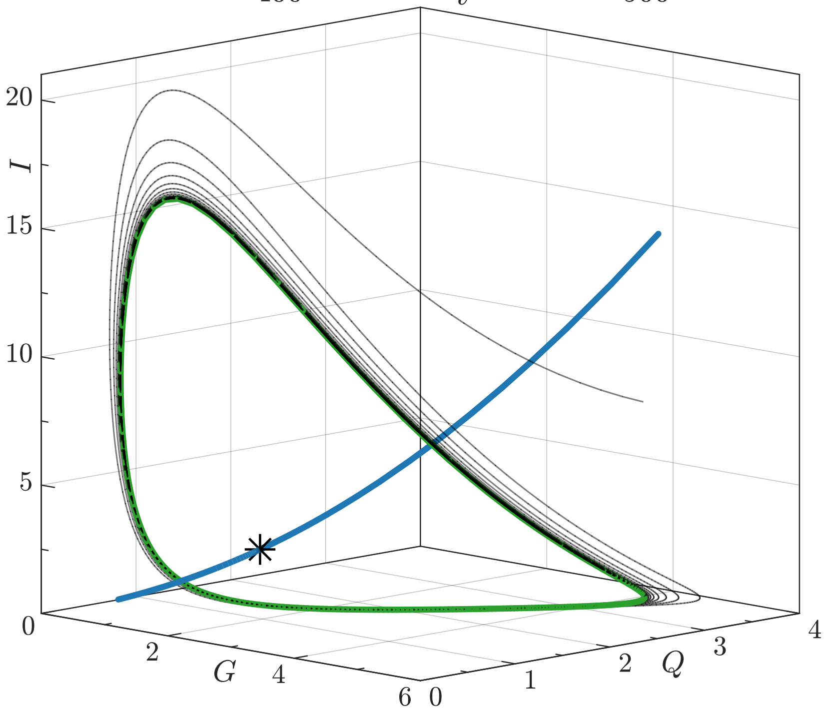
Continuation-based Computation of Global Isochrons*

Hinke M. Osinga† and Jeff Moehlis‡

PHASE-RESETTING: YAMADA MODEL



- Perturbations cause a phase shift ('lag') in intensity pulses.
- Phase difference $\approx \theta_{\text{old}} - \theta_{\text{new}}$
- Relationship between A_{p}, θ_{old}, and θ_{new}?



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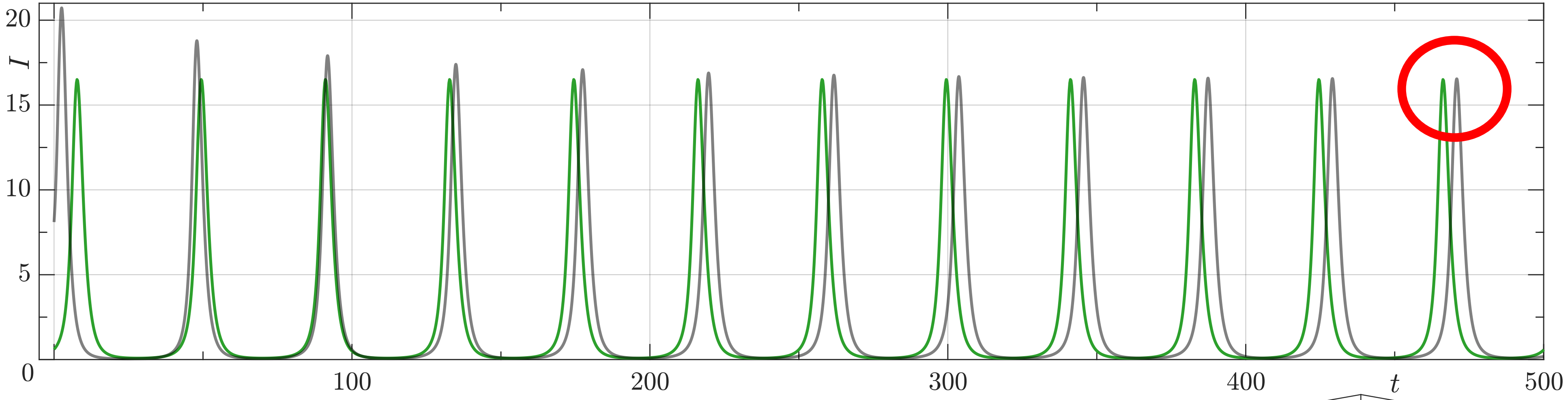

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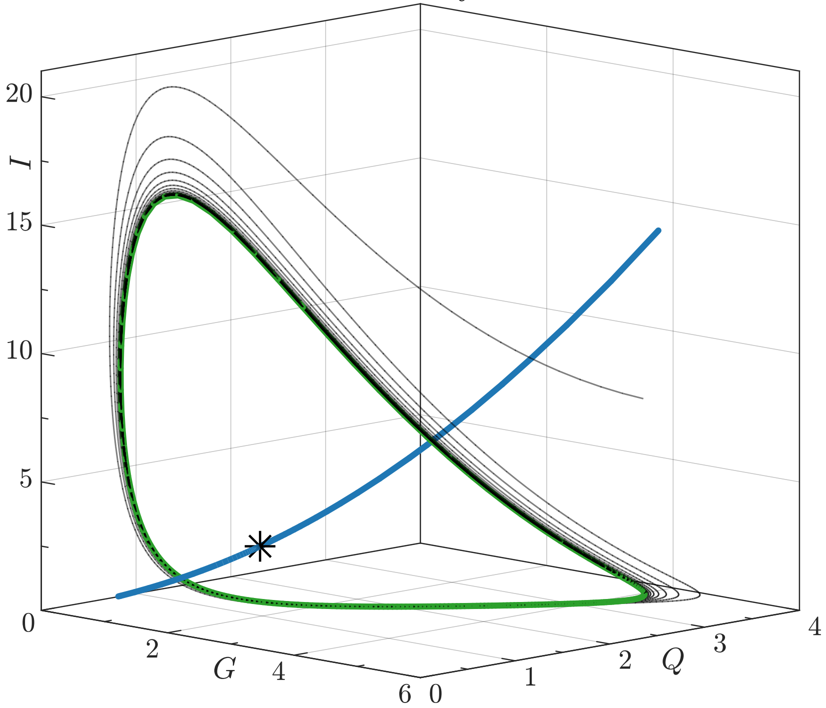
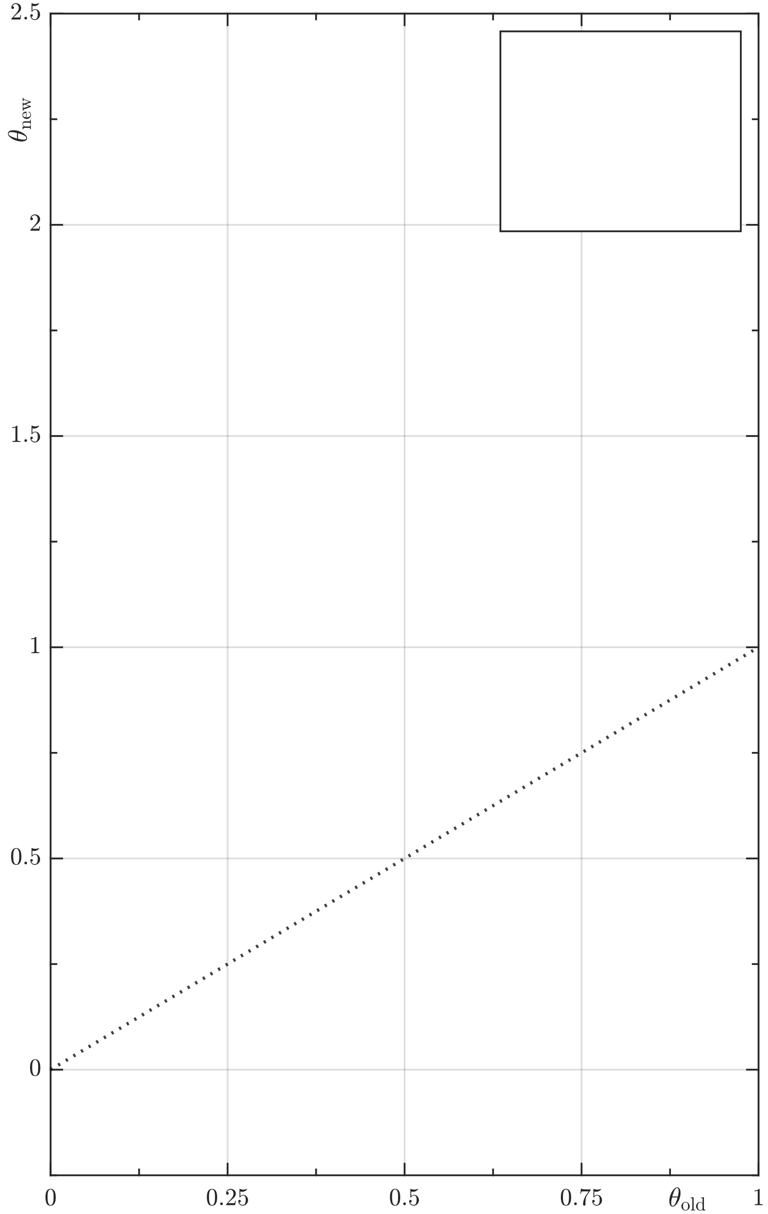


PHASE-TRANSITION CURVES (PTC)



- Positive-G perturbations
  - $d_{\mathrm{p}} = (0, 0, 1)$
- Weak perturbations “reset” to the same phase
  - $\theta_{\mathrm{p}} \approx \theta_{\mathrm{p}}$
- Stronger perturbations = ”difference”

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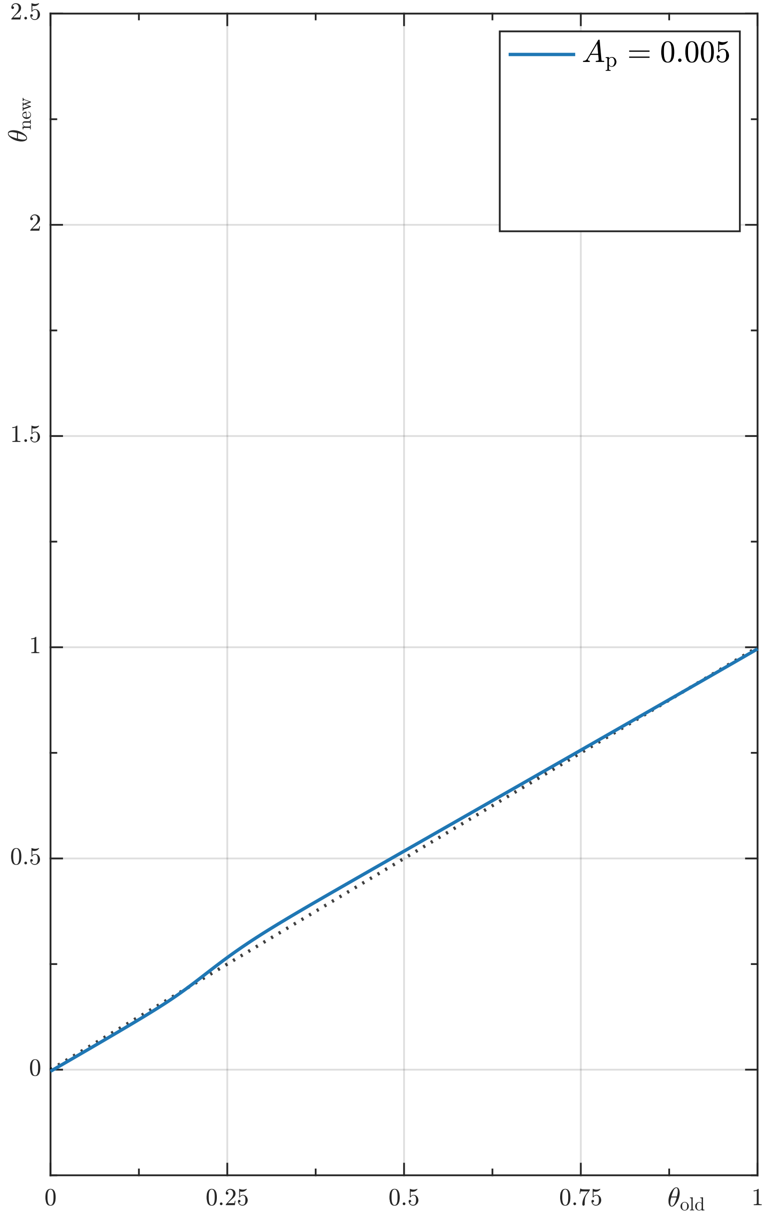

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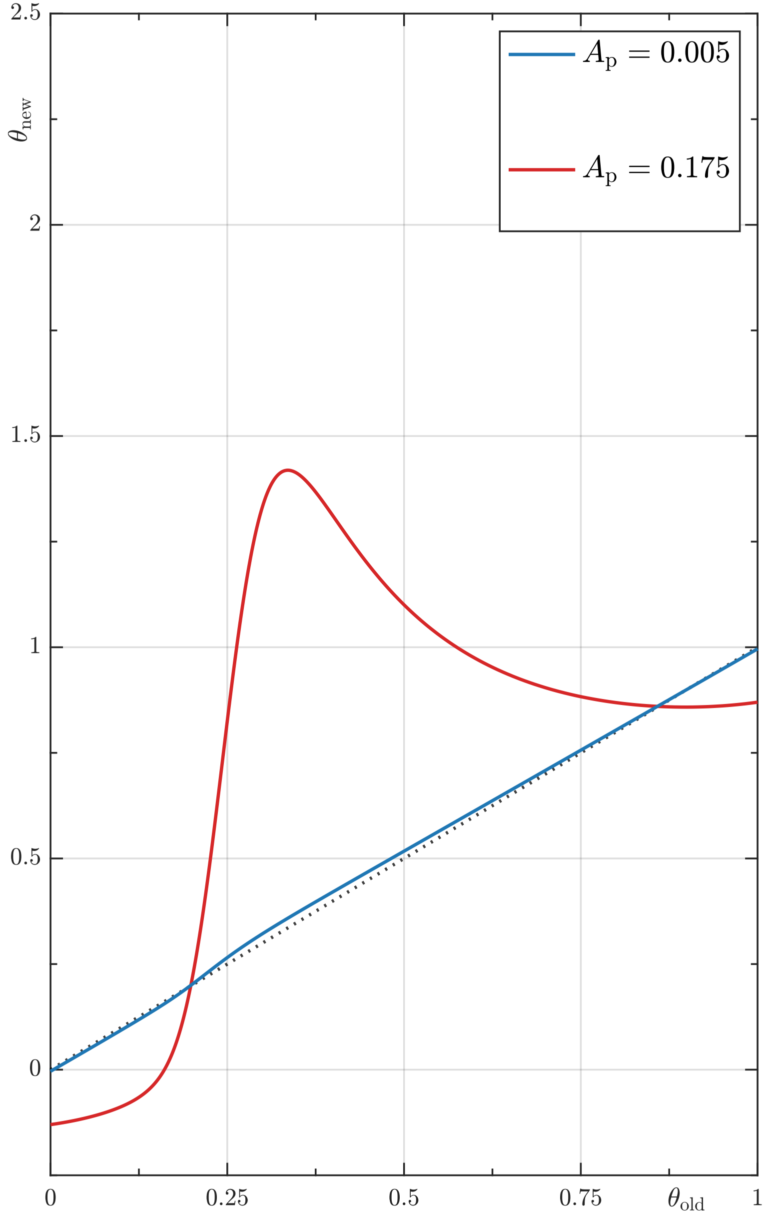

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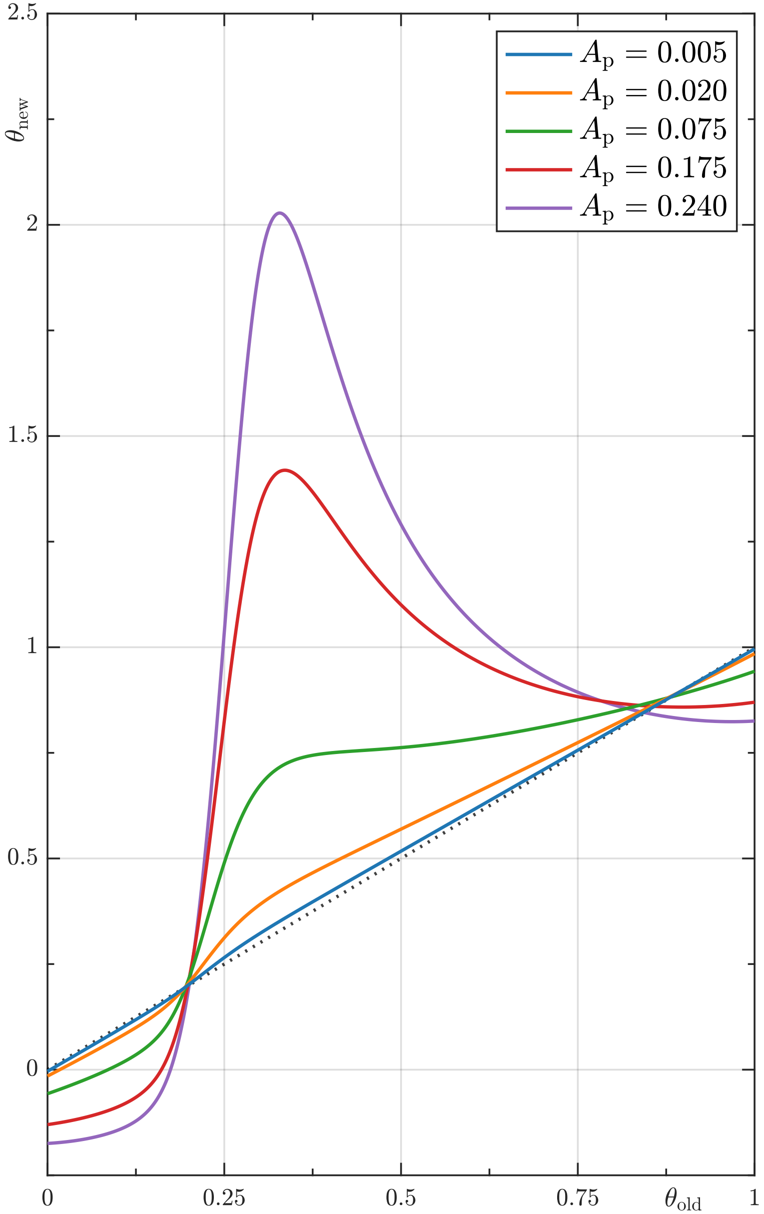

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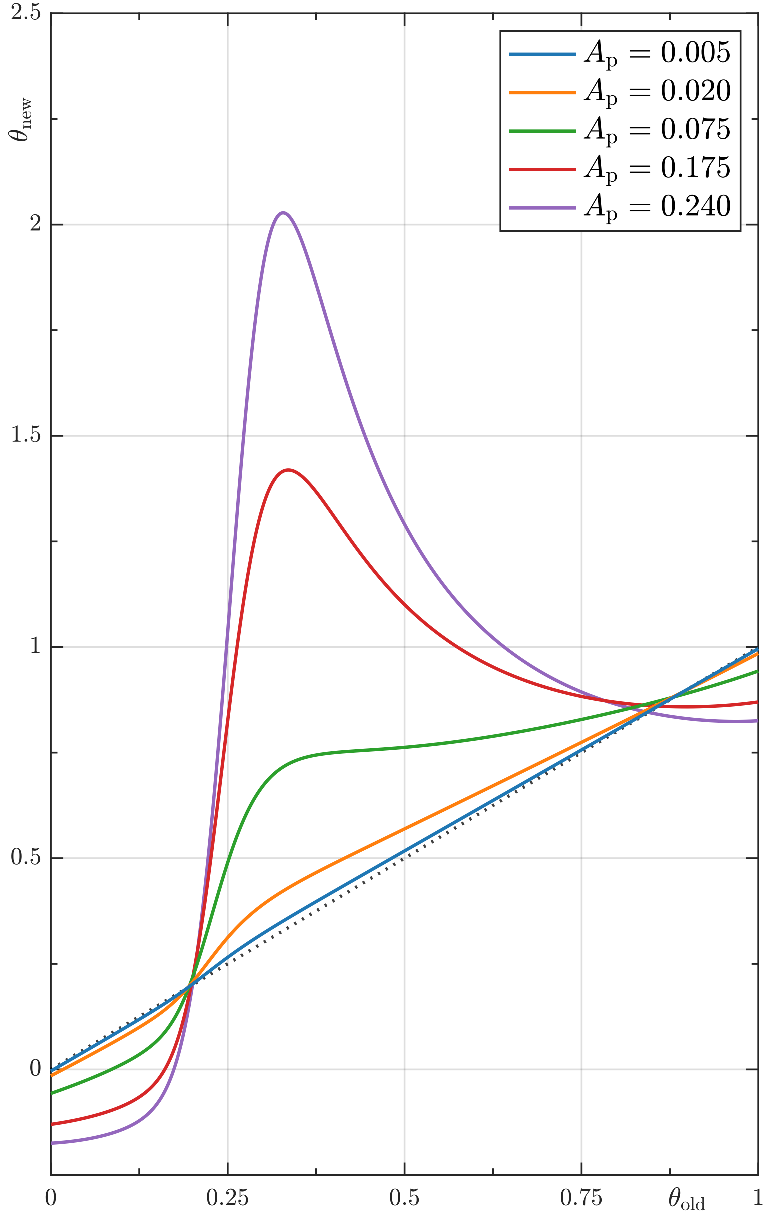

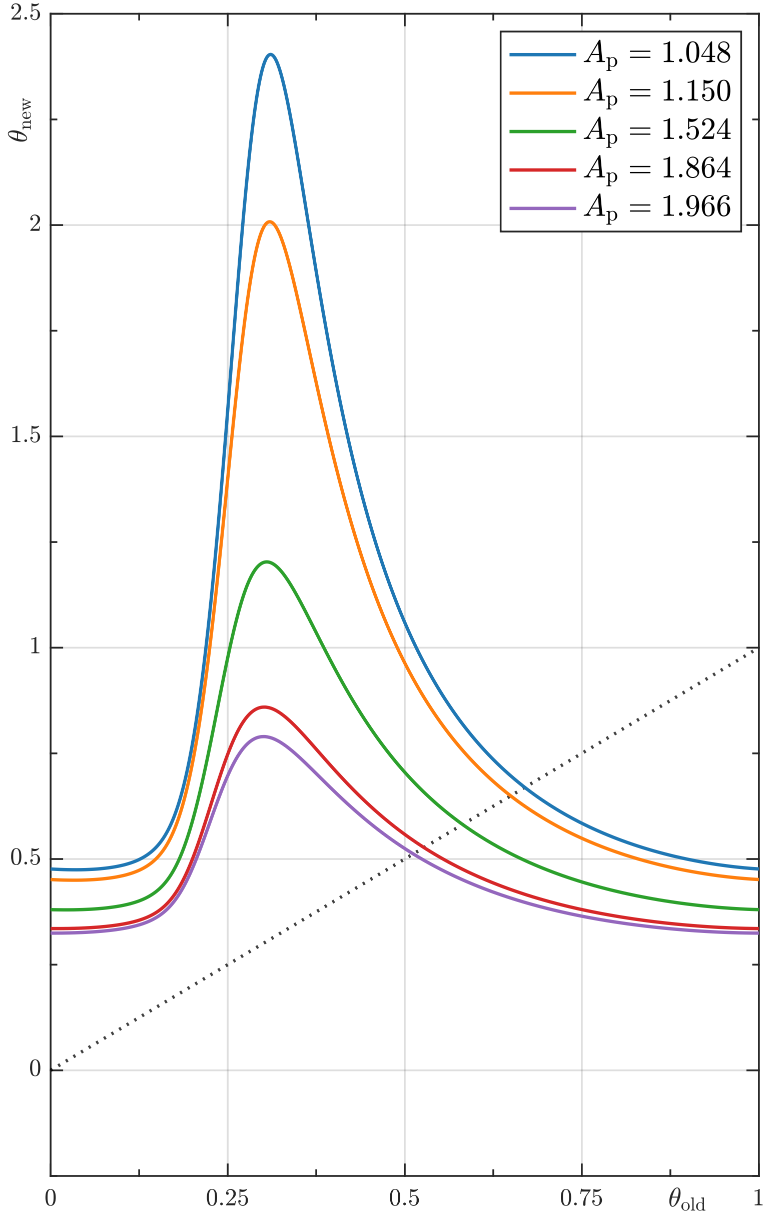

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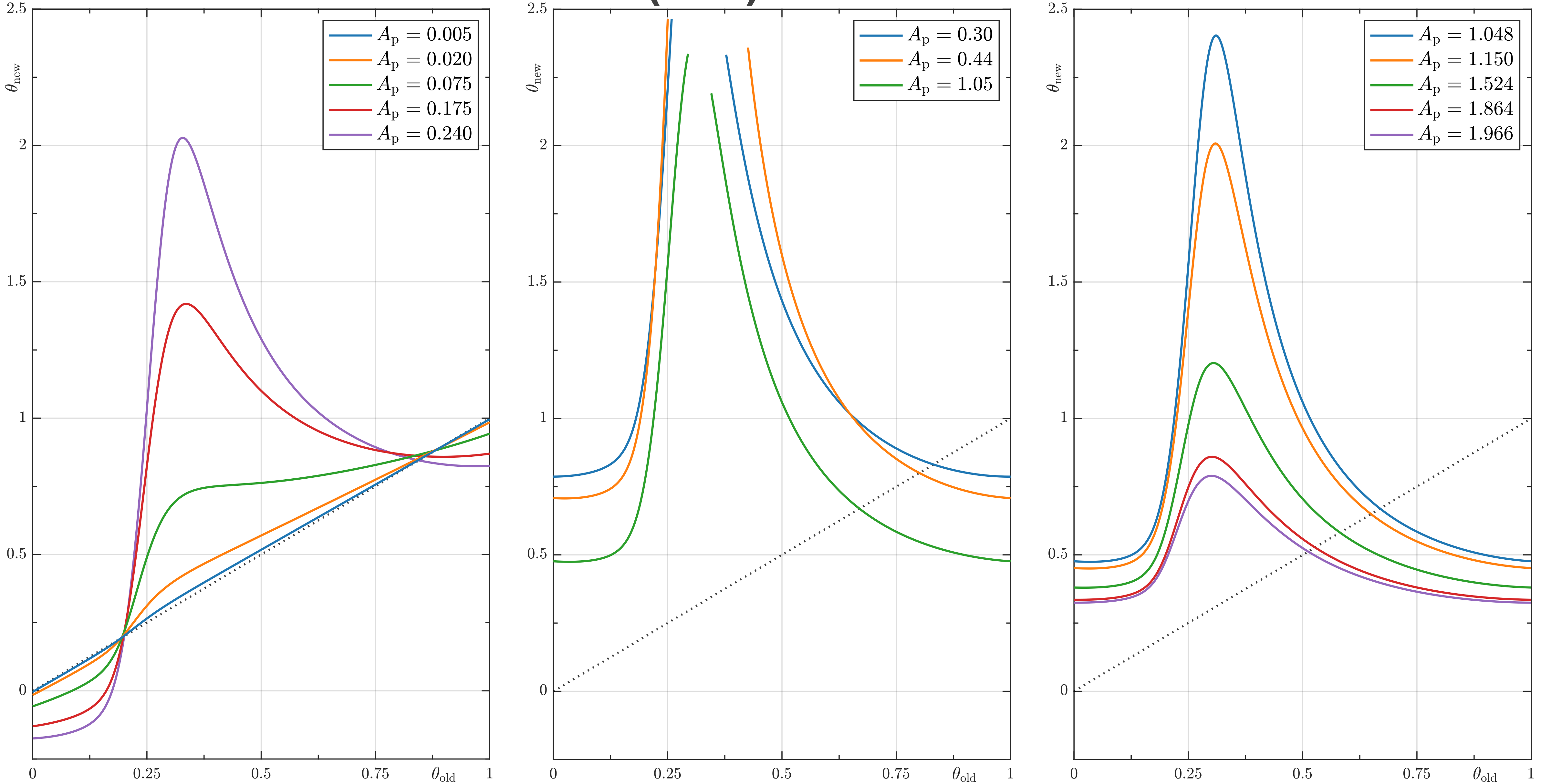

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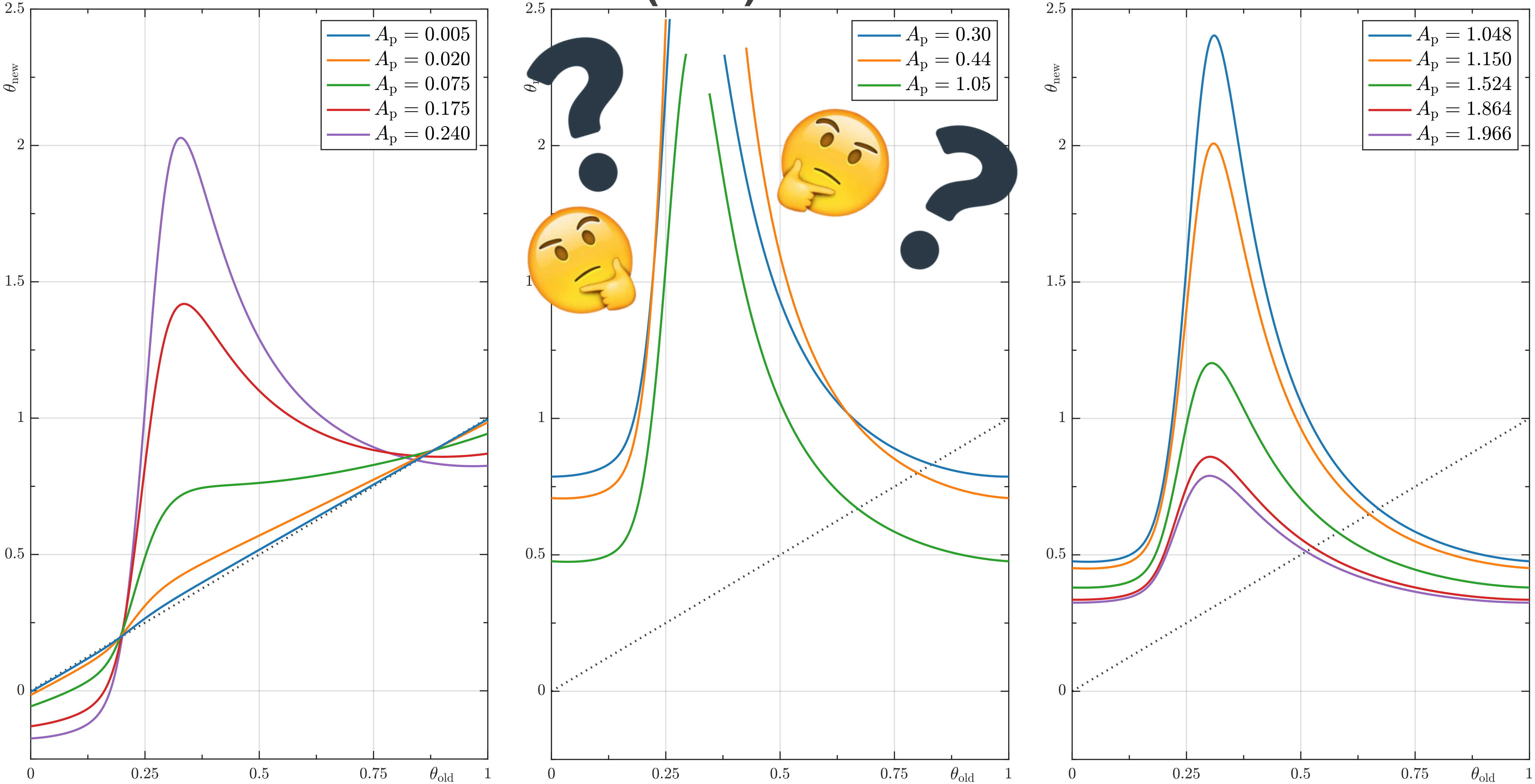
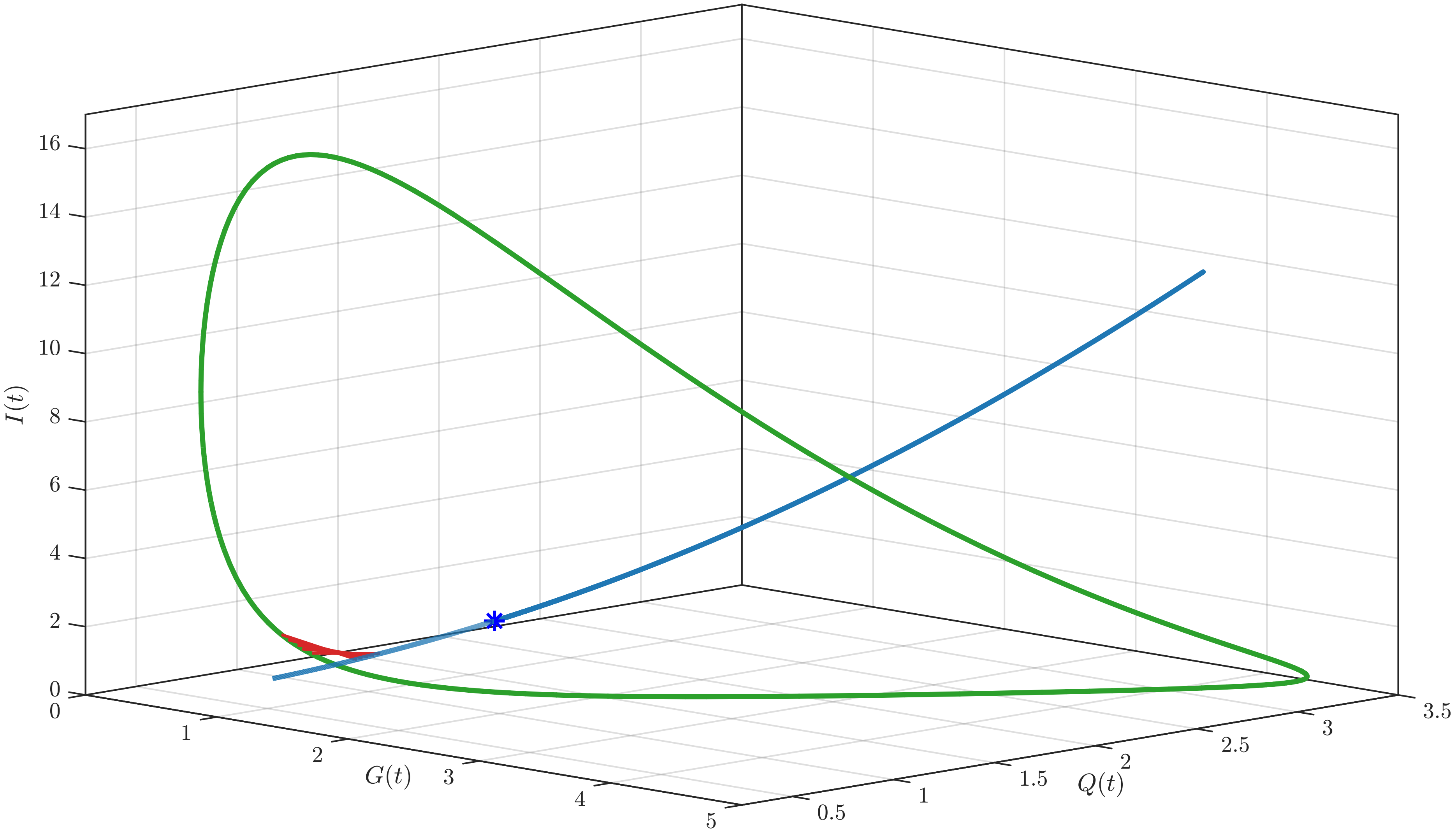


INTERSECTION WITH THE STABLE MANIFOLD

- Stable manifold of q intersects orbit $W^s(q)$
  - Initial point on stable manifold evolves towards q instead of “resetting”.
- Each point along orbit will have some perturbation pushing it into $W^s(q)$
  - Combination of $A_{\mathrm{p}}, d_{\mathrm{p}}$, and θ_{old}.
- Returned phase θ_{new} grows



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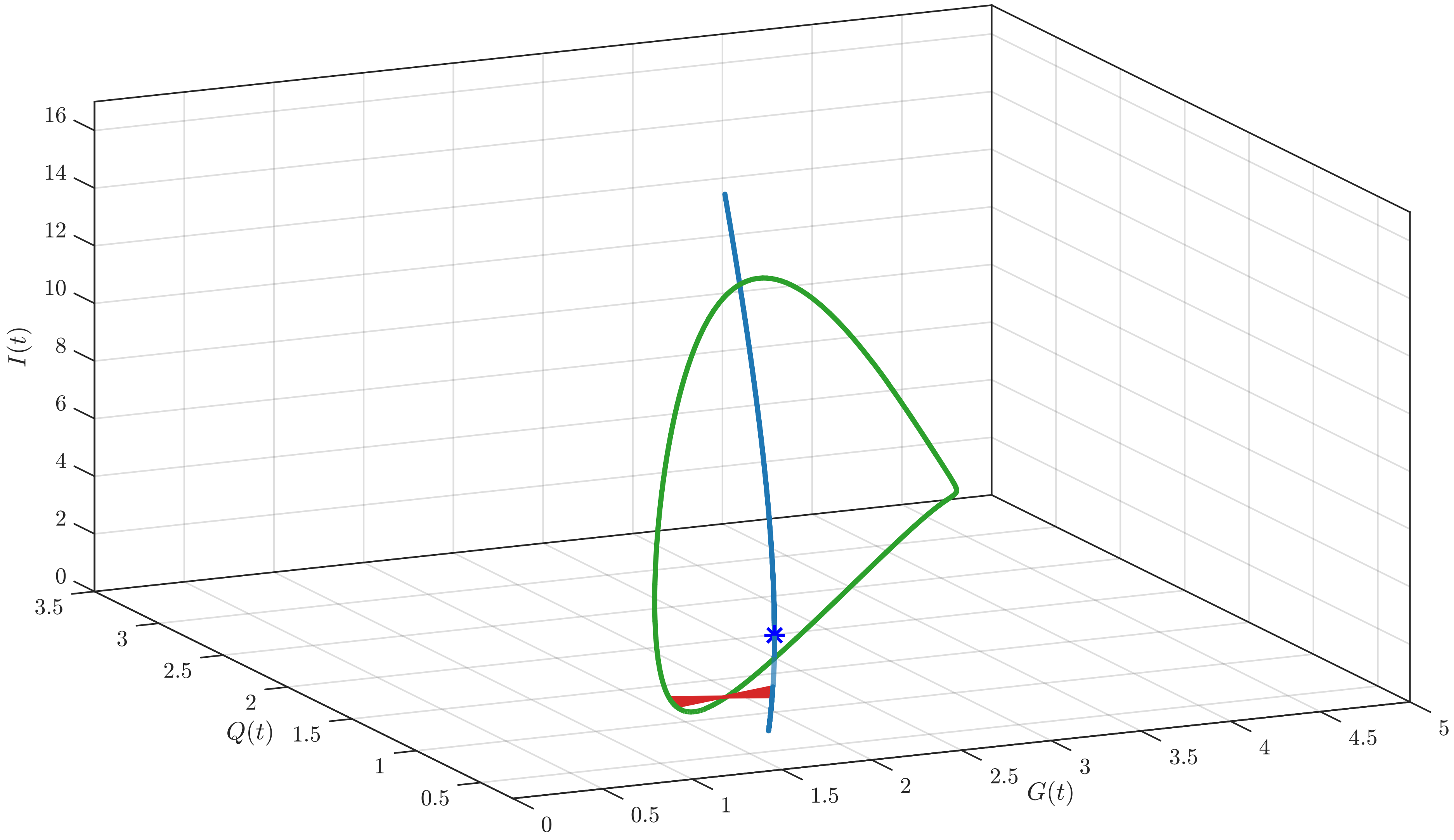

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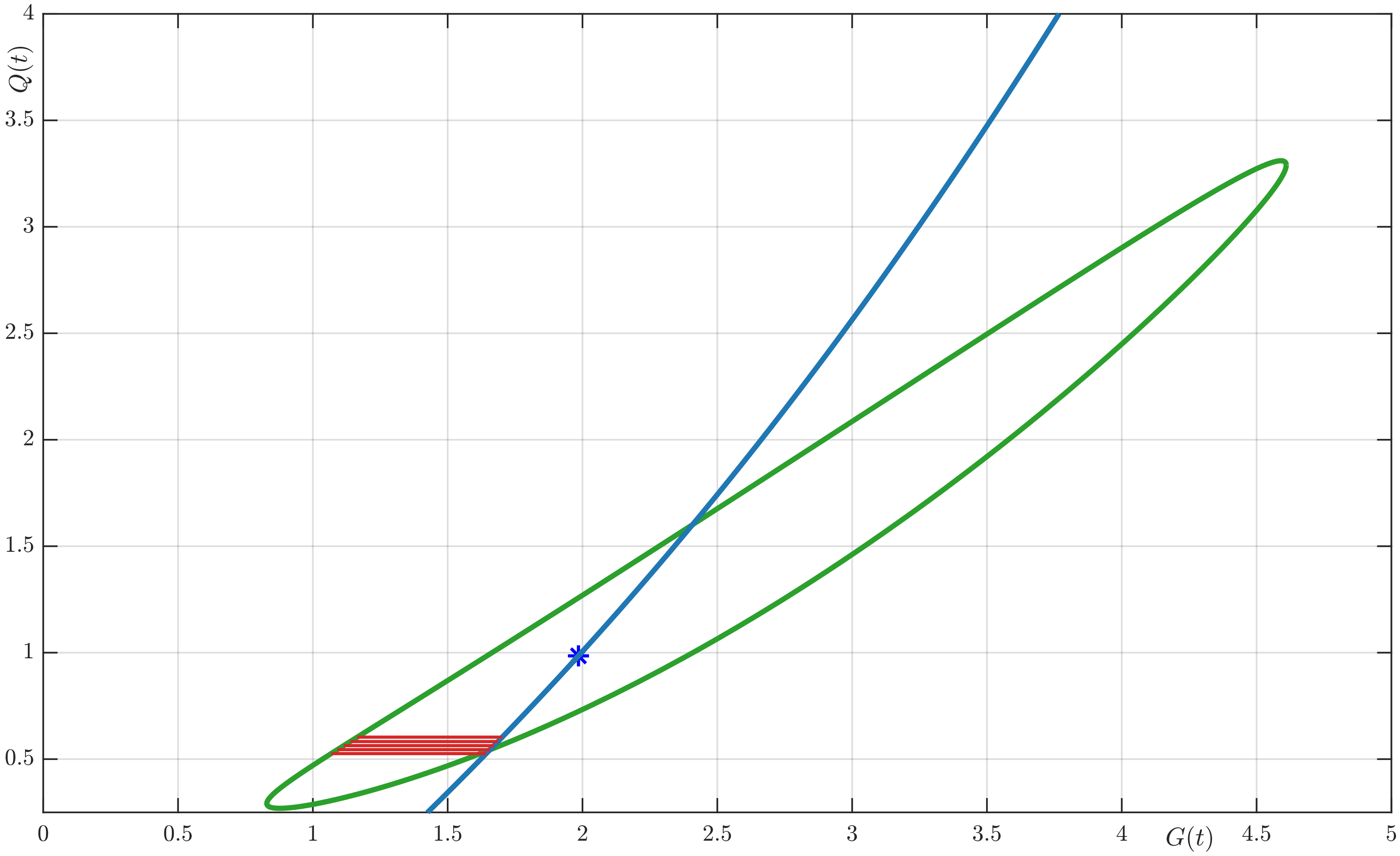
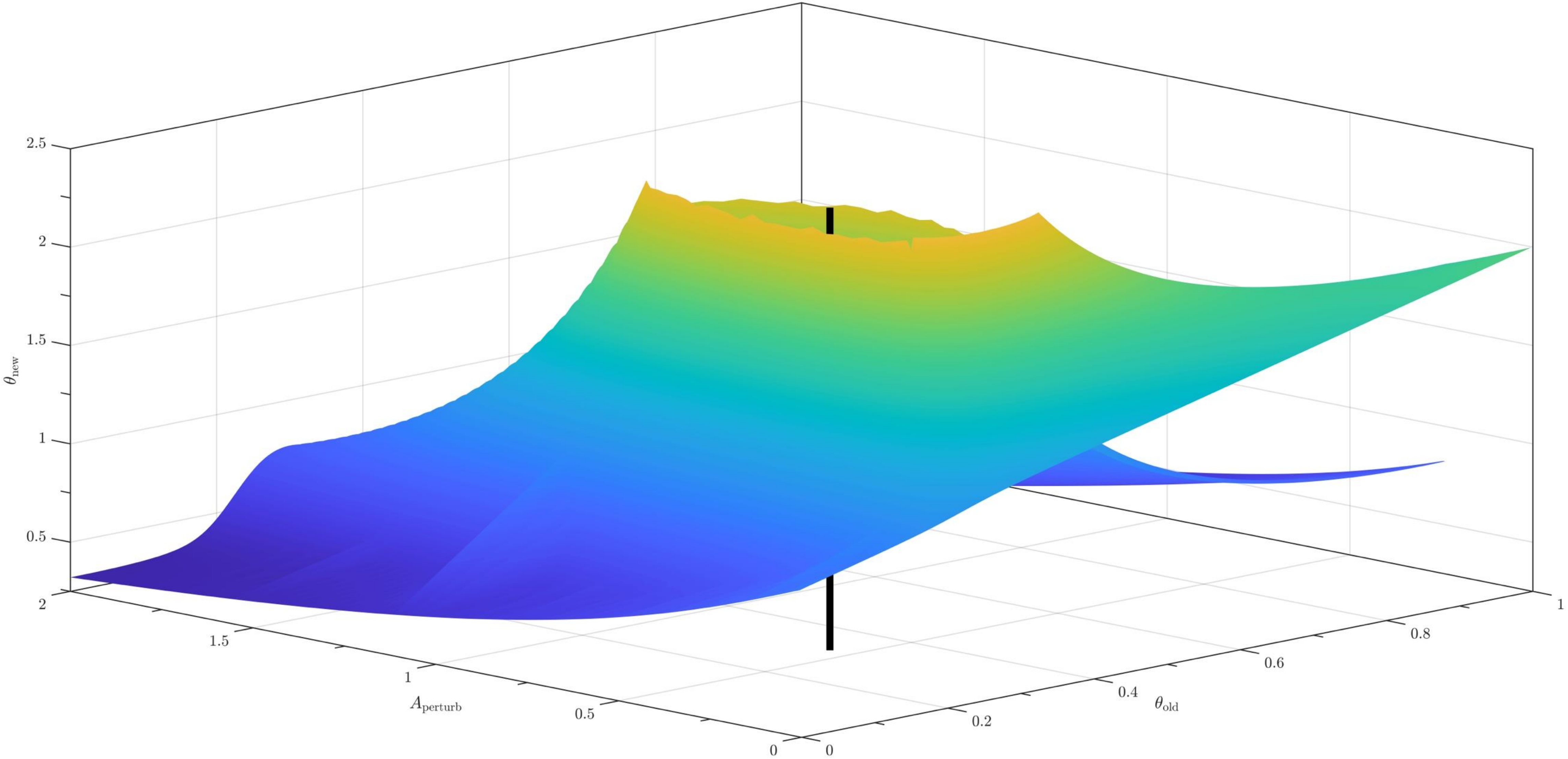
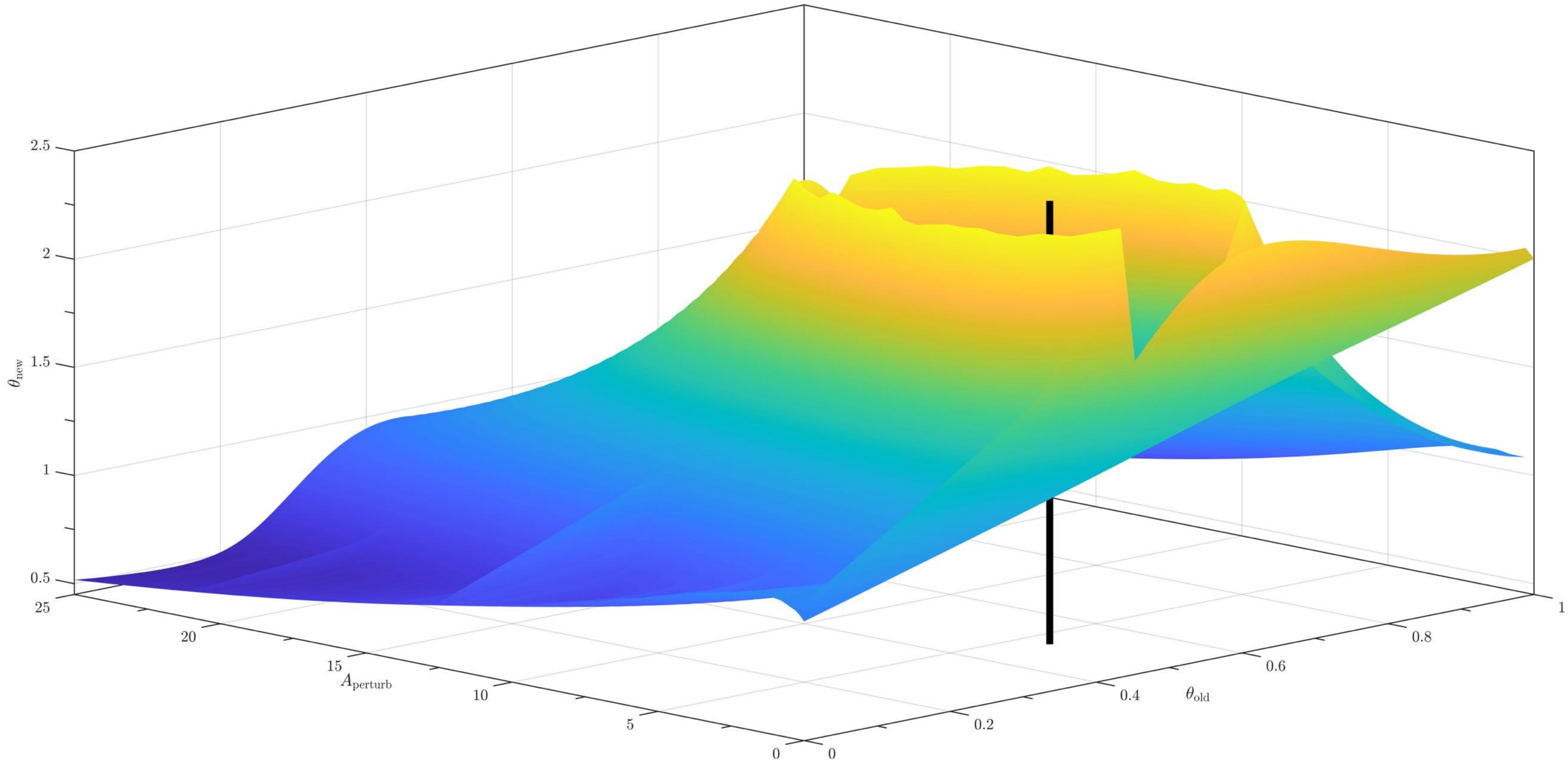


PTC SURFACE: *G*-PERTURBATION



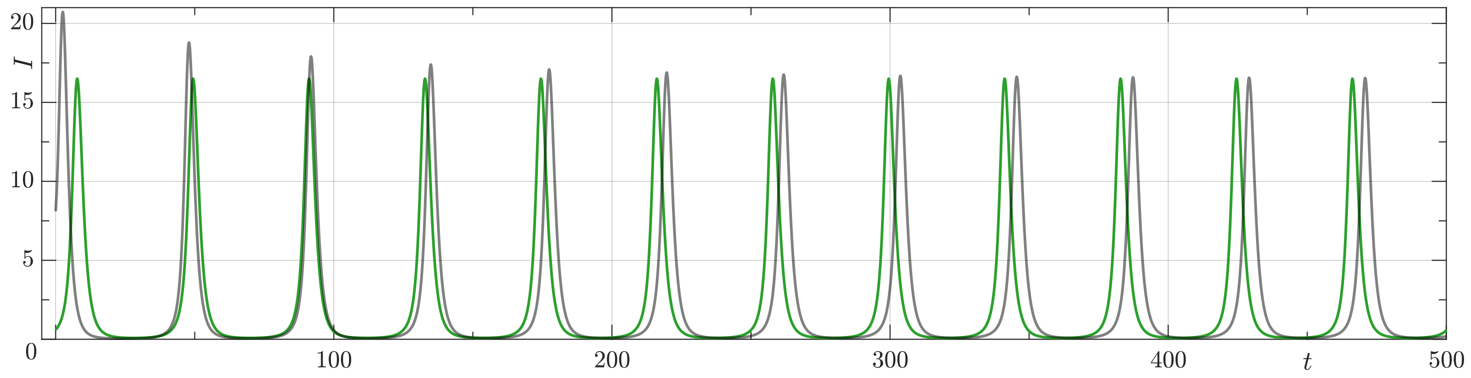
$d_{\mathrm{p}} = (1, 0, 0)$

PTC SURFACES: *I*-PERTURBATION



$d_{\mathrm{p}} = (0, 0, 1)$

CONCLUSIONS



- Phase-Resetting is a powerful tool in studying the response of periodic solutions to induced perturbations
- Discontinuities in PTC when perturbation approaches stable manifold of spiral source
- Can technically consider perturbation in any “direction”.